

On the uniqueness of Groves mechanisms*

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Abstract

We present a *necessary* and *sufficient* condition for Groves mechanisms to be unique among all possible efficient, dominant strategy direct mechanisms, in a social choice setting with quasi-linear preferences, multi-dimensional types, and private valuations. Our condition imposes restrictions on the behavior of the one-sided directional derivatives of the valuation functions with respect to individual types. We provide two simple examples to illustrate our main result, and carry out applications to the case of convex valuation functions and to Bayesian Nash environments, for which a payoff equivalence result is also proven.

Keywords: Dominant strategy mechanisms, Allocative efficiency, Groves mechanisms, Value function.

JEL Classification Numbers: C70, D70, D82.

1 Introduction

“The VCG analysis has become an important standard. It is the work by which nearly all other mechanism design work is judged and in terms of which its contribution is assessed.” (Milgrom [10], p. 45.)

Economists have known for some time that in social choice environments with quasi-linear preferences and private valuations, the class of mechanisms that sprang from the work of Vickrey [14], Clarke [1] and Groves [4], consists of dominant strategy incentive compatible mechanisms whose allocation rules select the efficient outcomes. That *any* direct mechanism satisfying those two properties is a member of the class of Groves (or VCG) mechanisms was first discovered by Green and Laffont [3]. Their result was later refined by Holmström [5],

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who presented sufficient conditions for the uniqueness of Groves mechanisms among all possible efficient, dominant strategy direct revelation mechanisms, based on differentiability properties of the valuation functions with respect to the individual types. Holmström [5] also provided an example in which the valuation function of an agent fails to be everywhere differentiable, with respect to his type; in this example, there exist dominant strategy, efficient mechanisms other than Groves mechanisms.

The uniqueness principle, when in place, is important in the following sense. Suppose the designer aims to investigate whether or not there exists any dominant strategy implementable mechanism that, in addition to allocative efficiency, satisfies other desirable properties. If Groves mechanisms exhaust the class of efficient, dominant strategy mechanisms, then invoking the uniqueness result, the search shall be restricted to verifying if there is some Groves mechanism for which the additional requirements hold. From Holmström [5], we have learnt that such approach is valid if the individual valuations are everywhere differentiable with respect to types.

It would be unfortunate if everywhere differentiability turned out to be necessary for the uniqueness result, since in an important range of economic applications this condition seems inadequate. Consider for instance the design of a mechanism to award a prize - an object, multiple objects, a franchise - to a group of agents who, after the allocation problem is resolved, can take a hidden action. Suppose that agent i has a finite set of actions, \mathcal{A}_i , and that his type θ_i affects the appraisal of the allocation x in a linear way. In this case, i 's valuation of the allocation x when his type is θ_i , is given by $v_i(x, \theta_i) = \sup_{a_i \in \mathcal{A}_i} f_i(x, \theta_i, a_i)$, where f_i is an affine function of θ_i . Since the valuation function v_i is itself the outcome of an optimization program, v_i will most certainly fail to be everywhere differentiable in types. Notwithstanding its economic significance, this case is not covered by Holmström [5] sufficient conditions. Necessary conditions for the uniqueness result, on the other hand, are so far absent in the literature.

The purpose of this note is to fill this void. We present a *necessary* and *sufficient* condition for the complete characterization of efficient, dominant strategy mechanisms in terms of Groves mechanisms. Our condition does not rely on everywhere differentiability of valuation functions with respect to types. We work in a social choice setting with quasi-linear preferences, private valuations and multi-dimensional convex type spaces. Each valuation function is assumed to be regular equi-Lipschitz continuous relative to the space of individual types. We deal with an arbitrary set of social alternatives or allocations, and do not impose restrictions on the behavior of the valuation functions with respect to the allocation set (but we assume an efficient allocation can always be selected). Rather, the focus is on the properties of the valuation functions with respect to the individual type spaces.

We start with the observation that the social problem of allocative efficiency imposes a restriction on the behavior of the one-sided directional derivatives of the valuation functions, with respect to individual types. We prove that the reverse condition, imposed at efficient allocations where the value function (max-function) of the social problem is differentiable, constitutes a necessary and sufficient condition for the uniqueness of Groves mechanisms among all possible efficient, dominant strategy direct mechanisms. Our main result has a simple yet precise interpretation: any efficient, dominant strategy incentive compatible

mechanism is a Groves mechanism if and only if at efficient allocations at which the value function of the social choice problem is differentiable, the individual valuation function is (Gâteaux) differentiable with respect to types, i.e., it admits two-sided directional derivatives. This restriction is not vacuous: one can construct parametric optimization problems where the value function is everywhere differentiable, yet the valuation function lacks two-sided directional derivatives at the optima.¹ This is the content of Section 2.

Section 3 contains two simple examples to illustrate our main result. The first one shows the *impossibility* of having efficient, dominant strategy direct mechanisms other than Groves mechanisms when the valuation functions (and the value function) are not everywhere differentiable, as long as the condition for our main result is in place. The second example pays attention to the *possibility* of having efficient, dominant strategy direct mechanisms other than Groves mechanisms when the valuation function of an agent is not differentiable with respect to types, at equilibrium points at which the relevant value function is differentiable. This is a reinterpretation of the example presented by Holmström [5].

In Section 4 we carry out extensions of our main result to auctions, and more generally to Bayesian Nash implementable mechanisms. The relationship between Groves mechanisms and Bayesian Nash mechanisms has been at the center of recent work in mechanism design.² In particular, Williams [15] has shown that in a social choice setting with independent private valuations, any efficient, Bayesian Nash incentive compatible revelation mechanism is payoff equivalent to some Groves mechanism, from an interim perspective (i.e., after each player has learned his type but before other types become observable). Thus, an extension of the Revelation Principle can be stated along these lines (Williams [15], p. 157): in any efficient Bayesian Nash equilibrium of any mechanism, the interim expected utility and transfer of an agent, as functions of his type, are the same as in a Groves mechanism (up to an additive constant).³ We point out that the uniqueness of Groves mechanisms among all possible dominant strategy, efficient revelation mechanisms is a necessary condition for this payoff equivalence principle. Otherwise, one could construct a non-Groves mechanism that is efficient and dominant strategy incentive compatible (and hence Bayesian Nash incentive compatible), but fails to be transfer-equivalent to a Groves mechanisms.

We present a version of the payoff equivalence principle for the private valuation case that concerns us (our version is closer to the one presented by Krishna and Maenner [8]). We then revise the extension of the Revelation Principle in the following terms: in any efficient, Bayesian Nash equilibrium of any mechanism, the interim expected utility and transfer of an agent are the same as in a Groves mechanism, only if the valuation function of such agent is differentiable at the equilibrium points at which the value function for social optimization problem is differentiable. Thus, compared to the standard Revelation Principle, the augmented version imposes further restrictions on the economic environment.

¹See Example 3.2 in Section 3.

²Milgrom [10], Jehiel and Moldovanu [7], Krishna and Perry [9] and Williams [15], among others, emphasized the relevance of Groves mechanisms as an important reference in mechanism design theory.

³To prove his result, Williams relies on the integral representation of the payoff function in Bayesian incentive compatible mechanisms. Different authors have stated sufficient conditions to get this integral representation. See especially Krishna and Maenner [8] and Milgrom and Segal [11].

2 Uniqueness of Groves Mechanisms

2.1 Preliminaries

We consider a general social choice problem in an economy composed of N agents, indexed by $i = 1, \dots, N$, a set \mathcal{X} of social alternatives or allocations - called the **allocation set** - and a private, transferable good (money). We deal with arbitrary allocation sets; that is, we shall not impose any topological or convexity restriction on \mathcal{X} .

The utility of each agent $i = 1, \dots, N$ is additively separable in the social alternative and the private good. Let then

$$u_i(x, \theta_i, t_i) = v_i(x, \theta_i) + t_i$$

denote agent i 's utility derived from alternative $x \in \mathcal{X}$ and quantity $t_i \in \mathbb{R}$ of the private good, when i 's type is θ_i . Types (signals) are private information; the type space of agent i is denoted by Θ_i , and is assumed to be an open, convex bounded subset of \mathbb{R}^{k_i} ($k_i \geq 1$). As is standard in the literature, we refer to the function $v_i : \mathcal{X} \times \Theta_i \rightarrow \mathbb{R}$ as the **valuation function** of agent i . Informational externalities are not permitted in this social choice setting; we restrict our attention to the case of private valuations.⁴

We assume that for every agent i the family of functions $\{v_i(x, \cdot) : \Theta_i \rightarrow \mathbb{R} \mid x \in \mathcal{X}\}$ is **equi-Lipschitzian relative to Θ_i** .⁵ That is to say, for each i we assume the existence of a non-negative number L_i such that:

$$|v_i(x, \theta_i) - v_i(x, \hat{\theta}_i)| \leq L_i \|\theta_i - \hat{\theta}_i\|, \quad \forall \theta_i, \hat{\theta}_i \in \Theta_i, \quad \forall x \in \mathcal{X}. \quad (1)$$

Condition (1) imposes a uniform Lipschitz restriction on the family of valuation functions of each agent i . If the set of allocations is finite, we can weaken the equi-Lipschitz assumption to a standard Lipschitz condition imposed on each function $\theta'_i \rightarrow v_i(x, \theta'_i)$, where $x \in \mathcal{X}$ (see Subsection 4.1).

We further assume that, for every agent i and every allocation x , the function $\theta'_i \rightarrow v_i(x, \theta'_i)$ is **regular on Θ_i** . Recall that for an open, convex set $Y \subset \mathbb{R}^k$ and a real-valued function g defined on Y , the one-sided directional derivative of g at $z \in Y$ in the direction $y \in \mathbb{R}^k$ is defined as the limit

$$D^+g(z; y) \equiv \lim_{\lambda \downarrow 0} \frac{g(z + \lambda y) - g(z)}{\lambda},$$

provided this limit exists. The function g is regular at $z \in Y$ if it admits one-sided directional derivatives at z in any direction $y \in \mathbb{R}^k$. The function g is then said to be regular on Y if it is regular at every z in Y .

⁴The case of interdependent valuations presents further complications. Williams and Radner [16] have shown that generally there is no dominant strategy, efficient mechanism in a model with two alternatives, two agents and interdependent valuations, even if signals are one dimensional. More recent possibility and impossibility results are summarized and discussed by Jehiel and Moldovanu [7].

⁵See Rockafellar [12], section 10.

The assumption of regularity imposed on valuation functions, combined with the equi-Lipschitz property, implies that for every i and every x in \mathcal{X} , the following limits exist and are finite:⁶

$$\begin{aligned} D^+ v_i(x, \theta_i; y) &\equiv \lim_{\lambda \downarrow 0} \frac{v_i(x, \theta_i + \lambda y) - v_i(x, \theta_i)}{\lambda}, & \forall y \in \mathbb{R}^{k_i}; \\ D^- v_i(x, \theta_i; y) &\equiv \lim_{\lambda \uparrow 0} \frac{v_i(x, \theta_i + \lambda y) - v_i(x, \theta_i)}{\lambda}, & \forall y \in \mathbb{R}^{k_i}. \end{aligned}$$

Regularity is a weaker assumption than it might seem at first. Since the function $\theta'_i \rightarrow v_i(x, \theta'_i)$ is Lipschitzian relative to Θ_i , it is differentiable almost everywhere on Θ_i (this follows from Rademacher's theorem). Thus, regularity requires that at points where differentiability fails, the function $v_i(x, \cdot)$ still admits finite one-sided directional derivatives. While it is not generally true that any Lipschitzian function is regular, the construction of such non-regular Lipschitzian function is not a trivial exercise.⁷ On the other hand, regularity is satisfied in different situations that have important economic applications; e.g. if g is convex on Y , then it is regular on Y .

The individual domain of preferences for social alternatives, denoted \mathcal{V}_i , is composed of all real-valued functions v_i defined on $\mathcal{X} \times \Theta_i$ that are regular equi-Lipschitzian relative to Θ_i . Any profile $v = (v_1, \dots, v_N)$ in $\mathcal{V} = \mathcal{V}_1 \times \dots \times \mathcal{V}_N$ is called an **admissible preference profile**. No assumption is made on properties of the valuation functions with respect to the allocation set; but we assume that a welfare maximizing allocation can always be selected (see expression (2) below). We mainly treat \mathcal{X} as an index set and exploit the properties of the collection of functions $\{v_i(x, \cdot) : \Theta_i \rightarrow \mathbb{R} \mid x \in \mathcal{X}\}$ on Θ_i .

A central planner uses a direct revelation mechanism to elicit private information and select a social alternative. Let Θ denote the Cartesian product of type spaces $\Theta_1, \dots, \Theta_N$. We also employ the standard notation Θ_{-i} to refer to the Cartesian product of all type spaces excluding that of agent i . A function $X : \Theta \rightarrow \mathcal{X}$ is called an **allocation rule**, and a function $t : \Theta \rightarrow \mathbb{R}^N$ is called a **transfer scheme**. A **direct mechanism** Γ is a pair (X, t) that, for any profile of announcements $\theta = (\theta_1, \dots, \theta_N)$ in Θ , selects an alternative $X(\theta)$ from \mathcal{X} and provides agents with net transfers $t(\theta) = (t_1(\theta), \dots, t_N(\theta))$ of the private good. A direct mechanism $\Gamma = (X^*, t)$ is said to be (weakly) **efficient** if the allocation rule $X^* : \Theta \rightarrow \mathcal{X}$ is efficient; i.e., if

$$X^*(\theta) \in \arg \max_{x \in \mathcal{X}} \sum_{j=1}^N v_j(x, \theta_j), \quad \forall \theta \in \Theta. \quad (2)$$

We shall assume throughout this paper that an efficient allocation $X^*(\theta)$ exists for all type profiles θ in Θ .⁸ A direct mechanism $\Gamma = (X, t)$ is said to be **dominant strategy** if for every agent i truth-telling is a dominant strategy; i.e., if

$$\theta_i \in \arg \max_{\theta'_i \in \Theta_i} v_i(X(\theta'_i, \theta_{-i}), \theta_i) + t_i(\theta'_i, \theta_{-i}), \quad \forall \theta \in \Theta, \quad \forall i = 1, \dots, N. \quad (3)$$

⁶Observe $D^- v_i(x, \theta_i; y) = -D^+ v_i(x, \theta_i; -y)$.

⁷See, for instance, Rockafellar [13], p. 97.

⁸If the allocation problem in (2) admits more than one solution, then any selection will do.

A direct mechanism $\Gamma^G = (X^*, t^G)$ is said to be a **Groves mechanism** if its allocation rule X^* is efficient and if for all $i = 1, \dots, N$, there exists a real-valued function \bar{h}_i defined on Θ_{-i} such that:

$$t_i^G(\theta) = \sum_{j \neq i} v_j(X^*(\theta), \theta_j) + \bar{h}_i(\theta_{-i}), \quad \forall \theta \in \Theta. \quad (4)$$

2.2 The uniqueness result

Let $\Gamma = (X^*, t)$ be an efficient, dominant strategy direct mechanism. For all $i = 1, \dots, N$, we define the real-valued function h_i on Θ by:

$$h_i(\theta) = t_i(\theta) - \sum_{j \neq i} v_j(X^*(\theta), \theta_j). \quad (5)$$

The following conditions must then be satisfied, for any type θ_i in Θ_i :

$$\theta_i \in \arg \max_{\theta'_i \in \Theta_i} v_i(X^*(\theta'_i, \theta_{-i}), \theta_i) + \sum_{j \neq i} v_j(X^*(\theta'_i, \theta_{-i}), \theta_j); \quad (6)$$

$$\theta_i \in \arg \max_{\theta'_i \in \Theta_i} v_i(X^*(\theta'_i, \theta_{-i}), \theta_i) + \sum_{j \neq i} v_j(X^*(\theta'_i, \theta_{-i}), \theta_j) + h_i(\theta'_i, \theta_{-i}). \quad (7)$$

Expression (6) follows from expression (2), the allocative efficiency condition. Expression (7) follows from (3), the dominant strategy implementability condition, once we make use of the function h_i defined in (5) to express transfers to agent i . To simplify notation, we write $\sum_{j \neq i} v_j(X^*(\theta'_i, \theta_{-i}), \theta_j) = v_{-i}(X^*(\theta'_i, \theta_{-i}), \theta_{-i})$. Given the efficient, dominant strategy direct mechanism $\Gamma = (X^*, t)$, and given a type profile θ_{-i} , i 's **value function of the social problem**, denoted by V_i^S , and i 's **value function of the individual problem**, denoted by V_i^I , are defined on Θ_i by the expressions:

$$V_i^S(\theta_i) = v_i(X^*(\theta), \theta_i) + v_{-i}(X^*(\theta), \theta_{-i}); \quad (8)$$

$$V_i^I(\theta_i) = v_i(X^*(\theta), \theta_i) + v_{-i}(X^*(\theta), \theta_{-i}) + h_i(\theta_i, \theta_{-i}). \quad (9)$$

Our first result deals with properties of these value functions. In particular, the equi-Lipschitz continuity of the individual valuation v_i with respect to its second argument implies that both value functions (max-functions) V_i^S and V_i^I are Lipschitzian relative to Θ_i , and therefore differentiable almost everywhere on Θ_i . Lemma 1 makes this claim precise.

Lemma 1. *Let $v = (v_1, \dots, v_N)$ be an admissible preference profile. Let $\Gamma = (X^*, t)$ be an efficient, dominant strategy direct mechanism. For any agent $i = 1, \dots, N$, the value functions V_i^S and V_i^I defined in (8) and (9) are Lipschitz continuous relative to Θ_i and differentiable almost everywhere on Θ_i .*

Proof See Appendix. ■

Using Lemma 1 one can also infer that the function h_i of (5) is Lipschitz continuous with respect to its first argument, and hence differentiable almost everywhere on Θ_i . Milgrom and Segal [11] used the alternative condition of absolute continuity of the valuation function with respect to a one-dimensional type space to deliver a result similar to Lemma 1. Since Lipschitzian functions are absolutely continuous, their result is more general than ours in the one-dimensional case. However, the extension of absolute continuity to multi-dimensional settings is not without difficulty. Lipschitz continuity, on the other hand, extends naturally to arbitrary normed linear spaces, and thus appears to be a more convenient assumption in models with multiple dimensions of types. The next result is the multi-dimensional analogue of a result also reported by Milgrom and Segal [11] in the one-dimensional case.

Lemma 2. *Let $v \in \mathcal{V}$ be an admissible preference profile, and let $\Gamma = (X^*, t)$ be an efficient, dominant strategy direct mechanism. If V_i^S is regular at $\theta_i \in \Theta_i$, then for any direction y in \mathbb{R}^{k_i} , it is the case that:*

$$D^+ v_i(X^*(\theta), \theta_i; y) \leq D^+ V_i^S(\theta_i; y); \quad (10)$$

$$D^- v_i(X^*(\theta), \theta_i; y) \geq D^- V_i^S(\theta_i; y). \quad (11)$$

Proof See Appendix. ■

Lemma 2 imposes restrictions on the behavior of the one-sided directional derivatives of the function $\theta'_i \rightarrow v_i(X^*(\theta), \theta'_i)$ at the efficient allocations at which V_i^S is differentiable, since in such case we have:

$$D^+ V_i^S(\theta_i; y) \equiv \lim_{\lambda \downarrow 0} \frac{V_i^S(\theta_i + \lambda y) - V_i^S(\theta_i)}{\lambda} = \lim_{\lambda \uparrow 0} \frac{V_i^S(\theta_i + \lambda y) - V_i^S(\theta_i)}{\lambda} \equiv D^- V_i^S(\theta_i; y).$$

Therefore, an immediate consequence of Lemma 2 is that, at each type θ_i at which V_i^S is differentiable, the left derivative of the valuation function v_i with respect to θ_i in the direction y , evaluated at $(X^*(\theta), \theta_i)$, is greater than its right counterpart:

$$D^- v_i(X^*(\theta), \theta_i; y) \geq D^+ v_i(X^*(\theta), \theta_i; y). \quad (12)$$

We emphasize that the restriction expressed in (12) is a consequence of the efficiency requirement of the direct revelation mechanism $\Gamma = (X^*, t)$. Property 1 below requires that the reverse inequality be satisfied at the efficient allocations at which the social value function V_i^S is differentiable at θ_i .

Property 1. *Given the admissible preference profile $v = (v_1, \dots, v_N)$ in \mathcal{V} , let $X^* : \Theta \rightarrow \mathcal{X}$ be an efficient allocation rule. Say that Property 1 is satisfied if for every agent $i = 1, \dots, N$, for each θ_{-i} in Θ_{-i} , and for each type θ_i in Θ_i for which V_i^S is differentiable at θ_i , it is the case that*

$$D^- v_i(X^*(\theta), \theta_i; y) \leq D^+ v_i(X^*(\theta), \theta_i; y), \quad \forall y \in \mathbb{R}^{k_i}. \quad (13)$$

When the space of types Θ_i is one-dimensional, Property 1 implies that the left derivative of the valuation function v_i with respect to i 's type, evaluated at $(X^*(\theta), \theta_i)$, is not greater than its right counterpart. Property 1 rules out downward kinks in the valuation function at equilibrium points at which the social value function is differentiable. If it is satisfied and V_i^S is differentiable at θ_i , then from expressions (12) and (13) we conclude that $D^+v_i(X^*(\theta), \theta_i; y) = D^-v_i(X^*(\theta), \theta_i; y)$ for any direction y . In such case, v_i admits two-sided directional derivatives with respect to its second argument, at $(X^*(\theta), \theta_i)$. This is the key element of the uniqueness result: Property 1 constitutes a necessary and sufficient condition for the complete characterization of efficient, dominant strategy incentive compatible direct mechanisms in terms of Groves mechanisms. The main result of this paper is the following theorem.

Theorem 1. *Let $v = (v_1, \dots, v_N) \in \mathcal{V}$ be an admissible preference profile. Any efficient, dominant strategy direct mechanism $\Gamma = (X^*, t)$ is a Groves mechanism if and only if Property 1 is satisfied.*

Proof For sufficiency, we adapt in general lines the argument of Holmström [5] to the case of Lipschitz valuation functions. Suppose that Property 1 is satisfied, and let $\Gamma = (X^*, t)$ be an efficient, dominant strategy mechanism. We shall show that it is indeed in the class of Groves. That is, we show that the function h_i , defined for Γ as in (5), is constant on Θ_i .

Fix a profile of types θ_{-i} belonging to Θ_{-i} . For all θ_i in Θ_i , we can write $h_i(\theta_i, \theta_{-i}) = V_i^I(\theta_i) - V_i^S(\theta_i)$. Thus $h_i(\cdot, \theta_{-i})$ is also Lipschitz continuous relative to Θ_i , and therefore differentiable almost everywhere in Θ_i . We claim that for each direction y in \mathbb{R}^{k_i} , the two-sided directional derivative of $h(\cdot, \theta_{-i})$ in the direction y , denoted by $Dh_i(\cdot, \theta_{-i}; y)$, is zero almost everywhere on Θ_i . This would imply that $h_i(\cdot, \theta_{-i})$ is constant on Θ_i . To show this claim fix an arbitrary direction y in \mathbb{R}^{k_i} , and define, for θ_i in Θ_i , the functions $\psi_{\theta_i}^S$ and $\psi_{\theta_i}^I$ on \mathbb{R} by:

$$\psi_{\theta_i}^S(\lambda) = v_i(X^*(\theta), \theta_i + \lambda y) + v_{-i}(X^*(\theta), \theta_{-i}) - V_i^S(\theta_i + \lambda y); \quad (14)$$

$$\psi_{\theta_i}^I(\lambda) = v_i(X^*(\theta), \theta_i + \lambda y) + v_{-i}(X^*(\theta), \theta_{-i}) + h_i(\theta_i, \theta_{-i}) - V_i^I(\theta_i + \lambda y). \quad (15)$$

Observe that $\psi_{\theta_i}^S(\lambda) \leq 0$ for any $\lambda \in \mathbb{R}$ and $\psi_{\theta_i}^S(0) = 0$. A similar assertion holds for $\psi_{\theta_i}^I$. Now, let θ_i be a type in Θ_i at which both value functions V_i^S and V_i^I are differentiable. Then Property 1 implies:

$$\begin{aligned} 0 &\leq \lim_{\lambda \uparrow 0} \frac{\psi_{\theta_i}^S(\lambda) - \psi_{\theta_i}^S(0)}{\lambda} = D^-v_i(X^*(\theta), \theta_i; y) - D^-V_i^S(\theta_i; y) \\ &\leq D^+v_i(X^*(\theta), \theta_i; y) - D^+V_i^S(\theta_i; y) \\ &= \lim_{\lambda \downarrow 0} \frac{\psi_{\theta_i}^S(\lambda) - \psi_{\theta_i}^S(0)}{\lambda} \leq 0. \end{aligned}$$

A similar argument holds if we use $\psi_{\theta_i}^I$ above instead. It follows readily that for almost

every θ_i in Θ_i , $\lim_{\lambda \rightarrow 0} \frac{\psi_{\theta_i}^S(\lambda)}{\lambda} = \lim_{\lambda \rightarrow 0} \frac{\psi_{\theta_i}^I(\lambda)}{\lambda} = 0$. Hence, almost everywhere in Θ_i :

$$Dh_i(\theta_i, \theta_{-i}; y) = \lim_{\lambda \rightarrow 0} \frac{h_i(\theta_i + \lambda y, \theta_{-i}) - h_i(\theta_i, \theta_{-i})}{\lambda} = \lim_{\lambda \rightarrow 0} \frac{\psi_{\theta_i}^S(\lambda) - \psi_{\theta_i}^I(\lambda)}{\lambda} = 0.$$

Since y was arbitrary, we conclude that for any direction y in \mathbb{R}^{k_i} , $Dh_i(\theta_i, \theta_{-i}; y) = 0$ almost everywhere on Θ_i . Since the function $h_i(\cdot, \theta_{-i})$ is Lipschitz continuous relative to Θ_i , it follows that $h_i(\cdot, \theta_{-i})$ is constant on Θ_i . Hence, $h_i(\theta) = \bar{h}_i(\theta_{-i})$, for any $\theta \in \Theta$. This shows that the mechanism $\Gamma = (X^*, t)$ is a Groves mechanism.

For necessity, suppose that for the preference profile $v = (v_1, \dots, v_N)$ in the domain \mathcal{V} , any efficient, dominant strategy direct mechanism $\Gamma = (X^*, t)$ is a Groves mechanism. Fix a type profile θ_{-i} in Θ_{-i} . Then it must be the case that the function $h_i(\cdot, \theta_{-i})$ of (9) is a constant function on Θ_i . Hence, for any θ_i and any direction y in \mathbb{R}^{k_i} , its two-sided directional derivative vanishes, i.e., $Dh_i(\theta_i, \theta_{-i}; y) = 0$.

Fix a direction y in \mathbb{R}^{k_i} and a type θ_i in Θ_i . Let $\psi_{\theta_i}^S$ and $\psi_{\theta_i}^I$ be as in (14) and (15). We show that these two functions are differentiable at $\lambda = 0$. Indeed, we have that

$$\begin{aligned} \psi_{\theta_i}^S(\lambda) - \psi_{\theta_i}^I(\lambda) &= V_i^I(\theta_i + \lambda y) - V_i^S(\theta_i + \lambda y) - h_i(\theta_i, \theta_{-i}) \\ &= h_i(\theta_i + \lambda y, \theta_{-i}) - h_i(\theta_i, \theta_{-i}) \\ &= 0. \end{aligned}$$

It follows that $\psi_{\theta_i}^S(\lambda) = \psi_{\theta_i}^I(\lambda)$ for any scalar λ . We may now write

$$\lim_{\lambda \rightarrow 0} \frac{h_i(\theta_i + \lambda y, \theta_{-i}) - h_i(\theta_i, \theta_{-i})}{\lambda} = \lim_{\lambda \rightarrow 0} \frac{\psi_{\theta_i}^S(\lambda) - \psi_{\theta_i}^I(\lambda)}{\lambda} = 0.$$

Thus, since $\psi_{\theta_i}^S(0) = \psi_{\theta_i}^I(0) = 0$, we conclude that the functions $\psi_{\theta_i}^S$ and $\psi_{\theta_i}^I$ are differentiable at $\lambda = 0$. At any point θ_i at which V_i^S is differentiable, the definition of $\psi_{\theta_i}^S$ in (14) implies that the function $\theta'_i \rightarrow v_i(X^*(\theta), \theta'_i)$ admits two-sided directional derivatives at θ_i in the direction y . Since y was again an arbitrary direction vector in \mathbb{R}^{k_i} , we conclude that Property 1 is satisfied. ■

3 Examples

In this section we provide two simple examples to illustrate our main result. The first example pays attention to the *impossibility* of having efficient, dominant strategy direct mechanisms other than Groves mechanisms when Property 1 is in place. It considers a social choice setting with one agent, a binary allocation set, and valuation functions, parameterized by allocations, that fail to be everywhere differentiable in types. The kink in each of the valuation function generates a kink in the value function of the social problem (which

here coincides with the value function of the individual problem). This feature does not prevent the uniqueness result, however, since at every point at which the value function is differentiable, the valuation function is itself differentiable.

The second is a reinterpretation of the example originally presented by Holmström [5]. It exhibits the *possibility* of having efficient, dominant strategy direct mechanisms different than Groves mechanisms when the valuation function of an agent is not differentiable with respect to types, at equilibrium points at which the social value function is differentiable. We revise Holmström's example⁹ in light of our main result.

3.1 Example

Consider a mechanism design problem with $N = 1$, a type space $\Theta_1 = [0, 1]$, and a discrete allocation set $\mathcal{X} = \{0, 1\}$. Agent 1's valuation function is indexed by $x \in \{0, 1\}$ as follows:

$$v_1(0, \theta_1) = \begin{cases} \frac{5}{8} - \theta_1 & : 0 \leq \theta_1 \leq \frac{1}{4}; \\ \frac{1}{2} - \frac{1}{2}\theta_1 & : \frac{1}{4} \leq \theta_1 \leq 1. \end{cases}$$

$$v_1(1, \theta_1) = \begin{cases} \frac{1}{2}\theta_1 & : 0 \leq \theta_1 \leq \frac{3}{4}; \\ \theta_1 - \frac{3}{8} & : \frac{3}{4} \leq \theta_1 \leq 1. \end{cases}$$

The efficient allocation rule selects $X^*(\theta_1) = 0$ if $0 \leq \theta_1 \leq \frac{1}{2}$, and $X^*(\theta_1) = 1$ if $\frac{1}{2} \leq \theta_1 \leq 1$. The value function of agent 1 has three kinks: at $\theta_1 = \frac{1}{4}$, $\frac{1}{2}$ and $\frac{3}{4}$; see Figure 1 below. Property 1 is however satisfied, since the kink in each function $\theta'_1 \rightarrow v_1(x, \theta'_1)$, $x = 0, 1$, coincides with a kink in the value function.

The mechanism $\Gamma^G = (X^*, t_1^G = \bar{h}_1)$ is a Groves mechanism (\bar{h}_1 is here a constant). Theorem 1 states that any efficient, dominant strategy direct mechanism must have this form. Suppose otherwise, and let $\hat{\Gamma} = (X^*, \hat{t}_1)$ be an efficient, dominant strategy incentive compatible mechanism, where \hat{t}_1 is some non-constant function defined on $[0, 1]$.

Fix $\theta_1 = \frac{1}{2}$ and let $X^*(\frac{1}{2}) = 0$ be a selection of the efficient allocation rule X^* . For any type $\theta'_1 < \frac{1}{2}$, the allocation rule determines $X^*(\theta'_1) = 0$. Using the incentive compatibility constraints one sees that $\hat{t}_1(\theta'_1) = \hat{t}_1(\frac{1}{2})$. Similarly, for any two distinct types θ'_1 and θ''_1 above $\frac{1}{2}$, the allocation rule selects $X^*(\theta'_1) = X^*(\theta''_1) = 1$, and thus the incentive compatibility constraints imply that $\hat{t}_1(\theta'_1) = \hat{t}_1(\theta''_1)$ any two such types. Let now $\theta'_1 \in (\frac{1}{2}, \frac{3}{4})$. A further use of the incentive constraint gives us the following equations:

$$v_1(X^*(\frac{1}{2}), \frac{1}{2}) + \hat{t}_1(\frac{1}{2}) - v_1(X^*(\theta'_1), \frac{1}{2}) - \hat{t}_1(\theta'_1) = \hat{t}_1(\frac{1}{2}) - \hat{t}_1(\theta'_1) \geq 0, \quad (16)$$

and

$$v_1(X^*(\theta'_1), \theta'_1) + \hat{t}_1(\theta'_1) - v_1(X^*(\frac{1}{2}), \theta'_1) - \hat{t}_1(\frac{1}{2}) = \theta'_1 - \frac{1}{2} + \hat{t}_1(\theta'_1) - \hat{t}_1(\frac{1}{2}) \geq 0. \quad (17)$$

As θ'_1 approaches $\frac{1}{2}$, we see from (17) that $\hat{t}_1(\theta'_1) - \hat{t}_1(\frac{1}{2}) \geq 0$. This and (16) together imply that $\hat{t}_1(\theta'_1) = \hat{t}_1(\frac{1}{2})$ for θ'_1 close enough to $\theta_1 = \frac{1}{2}$. We argued previously that transfers

⁹Notation has been changed to accommodate this paper's notation.

to all types above $\frac{1}{2}$ were equal, and thus we have that \hat{t}_1 is in fact a constant function on $[0, 1]$. Hence, $\hat{\Gamma}$ is a Groves mechanism.

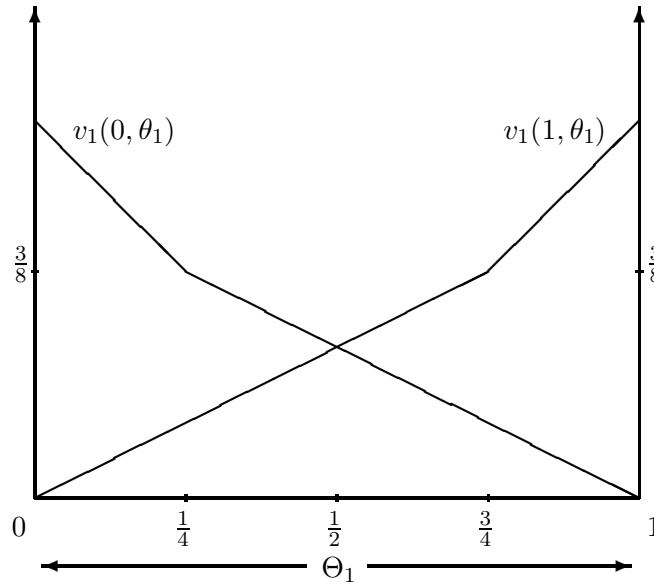


Figure 1

3.2 Example

Let now $N = 1, 2$. The allocation set is $\mathcal{X} = [0, 1]$; for each $i = 1, 2$, the type space is $\Theta_i = [0, 1]$. Preferences for social alternatives are determined by:

$$v_1(x, \theta_1) = \begin{cases} 0 & : \theta_1 \geq x, \\ \theta_1 - x & : \theta_1 \leq x; \end{cases}$$

and

$$v_2(x, \theta_2) = \theta_2 + \frac{1}{2}x.$$

In this example, the allocation rule $X^*(\theta) = \theta_1$ for any $\theta = (\theta_1, \theta_2)$ in Θ , is efficient. Observe that agent 1's value function of the social problem is $V_1^S(\theta_1) = \theta_2 + \frac{1}{2}\theta_1$, which is differentiable everywhere in $(0, 1)$. The following is a Groves mechanism: $\Gamma^G = (X^*, t_1^G, t_2^G)$, where the transfer scheme satisfies:

$$\begin{aligned} t_1^G(\theta) &= \theta_2 + \frac{1}{2}\theta_1 + \bar{h}_1(\theta_2); \\ t_2^G(\theta) &= \bar{h}_2(\theta_1). \end{aligned}$$

Consider now the alternative direct mechanism $\hat{\Gamma} = (X^*, \hat{t}_1, \hat{t}_2)$, with a transfer scheme (\hat{t}_1, \hat{t}_2) defined by:

$$\begin{aligned} \hat{t}_1(\theta) &= t_1^G(\theta) + \frac{1}{4}\theta_1; \\ \hat{t}_2(\theta) &= t_2^G(\theta). \end{aligned}$$

The mechanism $\hat{\Gamma}$ obviously does not belong to the class of Groves mechanisms. The reader can readily verify, however, that $\hat{\Gamma}$ is in effect an efficient, dominant strategy mechanism. The failure of the uniqueness of Groves mechanisms rests in that, at any efficient allocation $X^*(\theta) = \theta_1$ in $(0, 1)$, the value function V_1^S is differentiable but the left derivative of the function $\theta'_1 \rightarrow v_1(X^*(\theta), \theta'_1)$, evaluated at θ_1 , is strictly greater than its right counterpart (see Figure 2). This is a downward kink at efficient allocations, and constitutes a violation of Property 1, which is necessary to obtain the uniqueness result.

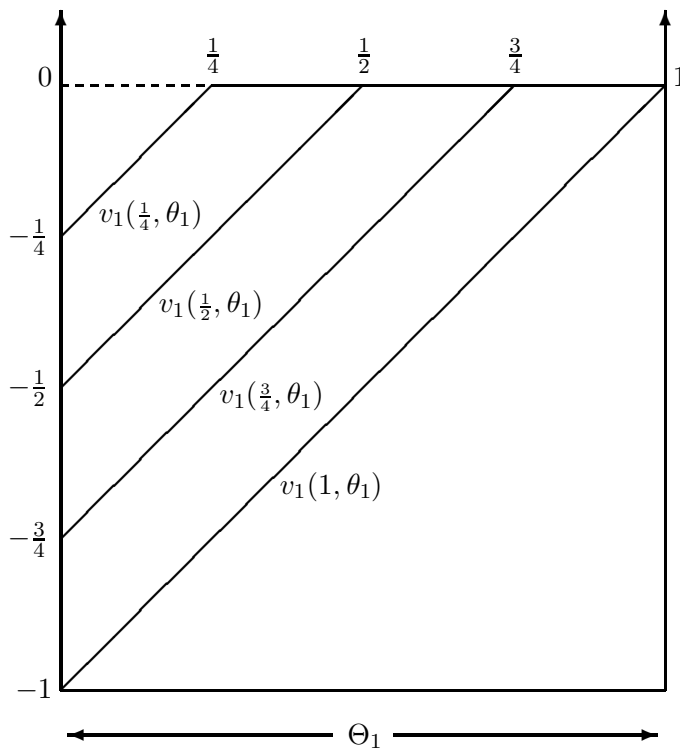


Figure 2

4 Extensions

4.1 An application to convex valuations

In a variety of interesting economic situations, auctions being a prominent example, the allocation set \mathcal{X} is finite. In such environments, Theorem 1 holds even if we weaken the equi-Lipschitz continuity condition imposed on valuation functions to a standard Lipschitz continuity on each of the functions $\theta'_i \rightarrow v_i(x, \theta'_i)$, for x in \mathcal{X} . Most models in auction theory work with individual valuation functions that are convex in types, and therefore locally Lipschitzian. With a finite allocation set, this convexity condition also implies Property 1, and

therefore is sufficient for the uniqueness result. In other common applications of mechanism design theory, the allocation set is infinite (e.g. the size of a public project, the design of a taxation system, etc.). With an arbitrary allocation set \mathcal{X} , convexity of the valuation functions with respect to types is not enough to guarantee the uniqueness result, since the equi-Lipschitzian condition may not be satisfied. However, convexity becomes sufficient with the inclusion of an appropriate boundedness condition, as we show next.

Say that the family of functions $\{v_i(x, \cdot) : \Theta_i \rightarrow \mathbb{R} \mid x \in \mathcal{X}\}$ is **pointwise bounded on** Θ_i if for every $\theta_i \in \Theta_i$, the set of real numbers $\{v_i(x, \theta_i)\}_{x \in \mathcal{X}}$ is bounded (see Rockafellar [12], section 10). We have the following result.

Theorem 2. *Let $v = (v_1, \dots, v_N)$ belong to \mathcal{V} and suppose that for any $i = 1, \dots, N$, $\{v_i(x, \cdot) : \Theta_i \rightarrow \mathbb{R} \mid x \in \mathcal{X}\}$ is a collection of pointwise bounded convex functions on Θ_i . Then any efficient, dominant strategy direct mechanism $\Gamma = (X^*, t)$ is a Groves mechanism.*

Proof See Appendix. ■

The well-known fact that in private valuation environments, any efficient auction format with truthful reporting as a dominant strategy is a VCG mechanism, shall be seen as a corollary of Theorem 2. Observe that this assertion applies to multi-unit or multi-object auctions with or without allocative externalities, as long as valuation functions are convex (but not necessarily everywhere differentiable) in types and there are no informational externalities.

4.2 A payoff-equivalence result for Bayesian Nash efficient mechanisms

Williams [15] has argued that a general equivalence result can be obtained in the independent private valuation case along the following lines. Consider any efficient revelation mechanism that is Bayesian Nash incentive compatible, i.e., truth telling constitutes a Bayesian Nash equilibrium strategy for all agents. Then from an interim perspective, that is after each player has learnt his type but before other types are known, the expected utility and transfer of each agent are the same as in some Groves mechanism. Thus, if one is after efficient, Bayesian Nash incentive compatible mechanisms that satisfy other desirable properties at an interim stage, one shall reduce the problem to verifying whether or not there exists some Groves mechanism that satisfies the additional properties.

At the heart of this augmented version of the Revelation Principle is the remarkable payoff equivalence result that has been reported by several authors, among others Jehiel and Moldovanu [6], Krishna and Maenner [8], Krishna and Perry [9], and Milgrom and Segal [11]. All aforementioned papers use some version of the integral representation of the expected payoff function to deliver this important result. We do so too, within the framework of the present work. Our version is closer to the one reported by Krishna and Maenner [8], whose work deals with valuation functions that are not everywhere differentiable in types, but are interdependent. To overcome the difficulties in the integral representation of the expected payoff function, Krishna and Maenner [8] assume that valuation functions and the mechanism are regular subdifferential (see below). Since we are preoccupied with the private

valuation case, we are able to drop the restriction on the type of mechanisms that are used in our version of the payoff equivalence principle. We next argue that, given the necessary conditions for the uniqueness of Groves mechanisms derived in Section 2, the augmented Revelation Principle requires more restrictions on the primitives than its original version.

Our model is as in Section 2, except that now we assume that types are independently distributed, each with a probability measure with full support. Some extra notation is needed. Given a direct mechanism $\Gamma = (X, t)$, denote agent i 's expected valuation and expected transfer, when his type is θ_i and his report is $\hat{\theta}_i$, and all other agents are truthfully reporting, by $\nu_i(\hat{\theta}_i, \theta_i)$ and $\tau_i(\hat{\theta}_i)$. We can write agent i 's expected utility, given type θ_i and report $\hat{\theta}_i$, as

$$\nu_i(\hat{\theta}_i, \theta_i) + \tau_i(\hat{\theta}_i) \equiv \mathbb{E}_{\theta_{-i}} \left[v_i(X(\hat{\theta}_i, \theta_{-i}), \theta_i) \right] + \mathbb{E}_{\theta_{-i}} \left[t_i(\hat{\theta}_i, \theta_{-i}) \right].$$

The direct mechanism $\Gamma = (X, t)$ is said to be **Bayesian Nash incentive compatible** if for every agent $i = 1, \dots, N$, truth-telling constitutes a Bayesian Nash equilibrium strategy; i.e. if for all i :

$$\theta_i \in \arg \max_{\hat{\theta}_i \in \Theta_i} \nu_i(\hat{\theta}_i, \theta_i) + \tau_i(\hat{\theta}_i), \quad \forall \theta_i \in \Theta_i.$$

Thus, given the direct revelation mechanism Γ , the **expected payoff function** of agent i is defined as the value function $U_i : \Theta_i \rightarrow \mathbb{R}$ that arises out of the Bayesian Nash incentive compatibility problem:

$$U_i(\theta_i) \equiv \nu_i(\theta_i, \theta_i) + \tau_i(\theta_i). \quad (18)$$

If U_i were continuously differentiable, then it could be represented as the integral of the derivative of ν_i with respect to its second argument. In general, however, the value function U_i will not be smooth. To overcome this difficulty, we follow the approach taken by Krishna and Maenner [8], and impose conditions on primitives that permit the application of tools from subdifferential calculus.

Let g be a regular Lipschitzian real-valued function defined on an open, convex set $Y \subset \mathbb{R}^k$. The **subdifferential of g at $z \in Y$** is the set define by

$$\partial g(z) \equiv \{s \in \mathbb{R}^k : s \cdot y \leq D^+ g(z; y), \quad \forall y \in \mathbb{R}^k\}.$$

A vector s of $\partial g(z)$ is called a subgradient of g at z . We make the observation that if g is differentiable at z then its gradient is the unique element in the subdifferential of g at z . Following F. H. Clarke [2], we also define the **generalized directional derivative of g at $z \in Y$ in the direction y** by:

$$D^\circ g(z; y) \equiv \limsup_{\substack{z' \rightarrow z \\ \lambda \downarrow 0}} \frac{g(z' + \lambda y) - g(z')}{\lambda};$$

and the **generalized subdifferential of g at z** by:

$$\partial^\circ g(z) \equiv \{s \in \mathbb{R}^k : s \cdot y \leq D^\circ g(z; y), \quad \forall y \in \mathbb{R}^k\}.$$

We know from Clarke [2] that if g is a Lipschitzian function, then the set $\partial^\circ g(z)$ is non-empty, for all z in the domain of g . In what follows, in addition to the assumptions made in Section 2, we shall assume that for each x in the allocation set \mathcal{X} the individual valuation function $v_i(x, \cdot)$ is **regular subdifferential** on Θ_i . That is to say, if x belongs to \mathcal{X} and θ_i belongs to $\Theta_i \subset \mathbb{R}^{k_i}$, then for any direction y in \mathbb{R}^{k_i} the one-sided directional derivative of $v_i(x, \cdot)$ at θ_i in the direction y coincides with the generalized directional derivative of $v_i(x, \cdot)$ at θ_i in the same direction. Hence, the function $v_i(x, \cdot)$ has a non-empty subdifferential everywhere in its domain, and it coincides with the generalized subdifferential set.

We now derive a payoff equivalence principle for Bayesian Nash mechanisms in the private valuation case.

Theorem 3. *Let $v \in \mathcal{V}$ be an admissible preference profile, and suppose in addition that each valuation function is regular subdifferential. Suppose that the direct mechanism $\Gamma = (X, t)$ is Bayesian Nash incentive compatible. Then for all $i = 1, \dots, N$, the expected payoff function U_i defined in (18) is determined by the allocation rule X up to an additive constant. We have:*

$$U_i(\theta_i) = U_i(\hat{\theta}_i) + \int_0^1 s(\lambda) \cdot (\theta_i - \hat{\theta}_i) d\lambda, \quad \forall \theta_i, \hat{\theta}_i \in \Theta_i; \quad (19)$$

where $\lambda \in (0, 1)$, $\theta_i(\lambda) = \lambda\theta_i + (1 - \lambda)\hat{\theta}_i$, and $s(\lambda)$ is a subgradient of the function $\theta'_i \rightarrow \nu_i(\theta_i(\lambda), \theta'_i)$ evaluated at $\theta_i(\lambda)$.

Proof See Appendix. ■

The meaning of (19) is the following. For any $\theta_i, \hat{\theta}_i$ in Θ_i , for every λ in $(0, 1)$, let $\theta_i(\lambda)$ be a point in the line segment connecting θ_i and $\hat{\theta}_i$, and let $s(\lambda)$ be any selection of the subdifferential set $\partial \nu_i(\theta_i(\lambda), \cdot)$ evaluated at $\theta_i(\lambda)$. Then the value of $U_i(\theta_i) - U_i(\hat{\theta}_i)$: is equal to $\int_0^1 s(\lambda) \cdot (\theta_i - \hat{\theta}_i) d\lambda$; is independent of the selection that one uses; depends only on i 's expected valuation and therefore depends only on the allocation rule X and not on transfers t_i .

An immediate consequence of Theorem 3 is that for any two Bayesian Nash incentive compatible mechanisms that share a certain allocation rule, say $\Gamma = (X, t)$ and $\hat{\Gamma} = (X, \hat{t})$, expected transfers to agents are equal up to a constant; i.e., for any $i = 1, \dots, N$:

$$\tau_i(\theta_i) - \hat{\tau}_i(\theta_i) = c_i, \quad \forall \theta_i \in \Theta_i.$$

In particular, the expected payoff of agent i generated by any efficient, Bayesian Nash incentive compatible revelation mechanism must be the same as the expected payoff generated by a Groves mechanism. We claim that our Property 1 is a necessary condition for this result. Suppose otherwise; since Property 1 is violated, it follows that Groves mechanisms are not unique among efficient, dominant strategy mechanisms. In such case, one could construct an efficient, dominant strategy (hence Bayesian Nash) incentive compatible mechanism that is not in the class of Groves.¹⁰ Let X^* denote the efficient allocation

¹⁰This was done, for instance, in Subsection 3.2.

rule and let $\Gamma^G = (X^*, t^G)$ be any Groves mechanism (where the transfer scheme t^G satisfies expression (4)). Consider now a non-Groves efficient, dominant strategy mechanism $\Gamma^{NG} = (X^*, t^{NG})$. Without loss of generality, we express the transfer scheme t^{NG} as

$$t_i^{NG}(\theta) = \sum_{j \neq i} v_j(X^*(\theta), \theta_j) + h_i(\theta_i, \theta_{-i}), \quad \forall \theta \in \Theta, \quad \forall i = 1, \dots, N;$$

where for each i , h_i is a real-valued function with domain Θ as defined in (5). Using expression (4), we see that

$$t_i^{NG}(\theta) - t_i^G(\theta) = h_i(\theta_i, \theta_{-i}) - \bar{h}_i(\theta_{-i}).$$

From the above equation it is clear that one could have a failure of the payoff equivalence principle, since it could be that for some i , the difference in expected transfers $\tau_i^{NG}(\theta_i) - \tau_i^G(\theta_i)$ is not constant.

Thus, we may now reinterpret the expanded version of the Revelation Principle proposed by Williams [15] in the following terms. If the payoff equivalence principle holds, then the expected payoffs in any efficient, Bayesian Nash implementable mechanism can be generated by a Groves mechanism only if the following condition holds: The individual valuation functions are differentiable with respect to types at all efficient allocations at which the value function that arises out of the social optimization problem (namely, the efficiency requirement) is differentiable with respect to types. Thus, compare to the standard version of the Revelation Principle, the augmented one imposes more restrictions on the economic problem.

Appendix

Proof of Lemma 1. Fix the type profile $\theta_{-i} \in \Theta_{-i}$ of all agents other than i , and consider any two distinct types θ_i and $\hat{\theta}_i$ in Θ_i . The definition of V_i^I in expression (9) implies that:

$$\begin{aligned} V_i^I(\theta_i) - V_i^I(\hat{\theta}_i) &= v_i(X^*(\theta), \theta_i) + v_{-i}(X^*(\theta), \theta_{-i}) + h_i(\theta_i, \theta_{-i}) \\ &\quad - v_i(X^*(\hat{\theta}_i, \theta_{-i}), \hat{\theta}_i) - v_{-i}(X^*(\hat{\theta}_i, \theta_{-i}), \theta_{-i}) - h_i(\hat{\theta}_i, \theta_{-i}) \\ &\leq v_i(X^*(\theta), \theta_i) + v_{-i}(X^*(\theta), \theta_{-i}) + h_i(\theta_i, \theta_{-i}) \\ &\quad - v_i(X^*(\theta), \hat{\theta}_i) - v_{-i}(X^*(\theta), \theta_{-i}) - h_i(\theta_i, \theta_{-i}) \\ &= v_i(X^*(\theta), \theta_i) - v_i(X^*(\theta), \hat{\theta}_i) \\ &\leq L_i \|\theta_i - \hat{\theta}_i\|; \end{aligned}$$

where the last inequality uses the equi-Lipschitz condition of the valuation functions. By reversing the roles of θ_i and $\hat{\theta}_i$ we obtain

$$V_i^I(\hat{\theta}_i) - V_i^I(\theta_i) \leq v_i(X^*(\hat{\theta}_i, \theta_{-i}), \hat{\theta}_i) - v_i(X^*(\hat{\theta}_i, \theta_{-i}), \theta_i) \leq L_i \|\theta_i - \hat{\theta}_i\|.$$

We conclude that for any two θ_i and $\hat{\theta}_i$ in Θ_i , $|V_i^I(\theta_i) - V_i^I(\hat{\theta}_i)| \leq L_i \|\theta_i - \hat{\theta}_i\|$. This shows that the value function of the individual problem, V_i^I , is Lipschitz continuous relative to Θ_i . A similar argument shows that the value function of the social problem defined in (8), V_i^S , is also Lipschitzian relative to its domain. Being Lipschitzian functions defined on an open convex set $\Theta_i \subset \mathbb{R}^{k_i}$, V_i^S and V_i^I are differentiable almost everywhere on their domain.¹¹ ■

Proof of Lemma 2. The definition of V_i^S in (8) implies that for any direction $y \in \mathbb{R}^{k_i}$ and any scalar λ sufficiently close to zero,

$$\begin{aligned} \left[v_i(X^*(\theta), \theta_i + \lambda y) + v_{-i}(X^*(\theta), \theta_{-i}) \right] - \left[v_i(X^*(\theta), \theta_i) + v_{-i}(X^*(\theta), \theta_{-i}) \right] \\ \leq V_i^S(\theta_i + \lambda y) - V_i^S(\theta_i). \end{aligned}$$

Thus, if $\lambda > 0$ we have:

$$\frac{v_i(X^*(\theta), \theta_i + \lambda y) - v_i(X^*(\theta), \theta_i)}{\lambda} \leq \frac{V_i^S(\theta_i + \lambda y) - V_i^S(\theta_i)}{\lambda}; \quad (20)$$

whereas if $\lambda < 0$ we have:

$$\frac{v_i(X^*(\theta), \theta_i + \lambda y) - v_i(X^*(\theta), \theta_i)}{\lambda} \geq \frac{V_i^S(\theta_i + \lambda y) - V_i^S(\theta_i)}{\lambda}. \quad (21)$$

If V_i^S is regular at θ_i , letting $\lambda \downarrow 0$ in (20) we obtain (10), and similarly letting $\lambda \uparrow 0$ in (21) we obtain (11). ■

Proof of Theorem 2. The convexity of each function $\theta'_i \rightarrow v_i(x, \theta'_i)$ implies that for each alternative x in \mathcal{X} , the function $\theta'_i \in \Theta_i \rightarrow v_i(x, \theta'_i)$ has one-sided directional derivatives everywhere in Θ_i . Moreover, for all x in \mathcal{X} , for any direction y in \mathbb{R}^{k_i} , it is the case that $D^- v_i(x, \theta_i; y) \leq D^+ v_i(x, \theta_i; y)$. It follows that our Property 1 is satisfied.

On the other hand, the fact that the collection $\{v_i(x, \cdot) : \Theta_i \rightarrow \mathbb{R} \mid x \in \mathcal{X}\}$ is pointwise bounded and convex on Θ_i guarantees that it is equi-Lipschitzian relative to any closed subset of Θ_i (Rockafellar [12], p. 88). Since for all i , the equi-Lipschitzian property fails at most on a subset of Θ_i of arbitrary small measure, we can apply an argument similar to that in Theorem 1 to show that any efficient, dominant strategy direct mechanism is a Groves mechanism. ■

¹¹This assertion is a direct application of Rademacher's Theorem, which states that if the function $g : \mathbb{R}^k \rightarrow \mathbb{R}$ is Lipschitz continuous, then it is differentiable almost everywhere in \mathbb{R}^k . The interested reader is referred to Ziemer [17] (section 2.2) for a complete presentation of Rademacher's theorem.

Proof of Theorem 3. Suppose that the direct mechanism $\Gamma = (X, t)$ is Bayesian Nash incentive compatible. We first claim that the family of functions $\{\nu_i(\theta'_i, \cdot) : \Theta_i \rightarrow \mathbb{R} \mid \theta'_i \in \Theta_i\}$ is regular subdifferential equi-Lipschitzian relative to Θ_i . To see this, fix any announcement θ'_i . Then for any two distinct θ_i and $\hat{\theta}_i$ in Θ_i we have

$$\begin{aligned} |\nu_i(\theta'_i, \theta_i) - \nu_i(\theta'_i, \hat{\theta}_i)| &= |\mathbb{E}_{\theta_{-i}} [v_i(X(\theta'_i, \theta_{-i}), \theta_i) - v_i(X(\theta'_i, \theta_{-i}), \hat{\theta}_i)]| \\ &\leq \mathbb{E}_{\theta_{-i}} \left[|v_i(X(\theta'_i, \theta_{-i}), \theta_i) - v_i(X(\theta'_i, \theta_{-i}), \hat{\theta}_i)| \right] \\ &\leq \mathbb{E}_{\theta_{-i}} \left[L_i \|\theta_i - \hat{\theta}_i\| \right] = L_i \|\theta_i - \hat{\theta}_i\| , \end{aligned}$$

where $L_i \geq 0$ is the Lipschitz constant of the family of equi-Lipschitzian valuation functions $\{v_i(x, \cdot) : \Theta_i \rightarrow \mathbb{R} \mid x \in \mathcal{X}\}$. The above stream of inequalities does not depend on θ'_i , and therefore we conclude that the family $\{\nu_i(\theta'_i, \cdot) : \Theta_i \rightarrow \mathbb{R} \mid \theta'_i \in \Theta_i\}$ is equi-Lipschitzian relative to Θ_i . The regular subdifferential property follows from the fact that the integral of a family of regular subdifferential Lipschitzian functions is also regular subdifferential (Clarke [2], p. 76).

We next show that the value function U_i is regular subdifferential Lipschitzian relative to Θ_i . Observe that for any two $\theta_i, \hat{\theta}_i$ in Θ_i , the definition of U_i implies that

$$\begin{aligned} |U_i(\theta_i) - U_i(\hat{\theta}_i)| &\leq \sup_{\theta'_i} |\nu_i(\theta'_i, \theta_i) - \nu_i(\theta'_i, \hat{\theta}_i)| \\ &\leq \sup_{\theta'_i} L_i \|\theta_i - \hat{\theta}_i\| = L_i \|\theta_i - \hat{\theta}_i\| , \end{aligned}$$

which shows that the function U_i is Lipschitz continuous relative to Θ_i . A theorem of Clarke [2] (p. 86) can be applied to show that the value function U_i is regular subdifferential.

We proceed to show that expression (19) holds. A consequence of U_i being Lipschitz regular subdifferential on Θ_i is the existence, for any θ_i and $\hat{\theta}_i$ in Θ_i , of a number $\lambda \in (0, 1)$ and a vector $\theta_i(\lambda) = \lambda\theta_i + (1 - \lambda)\hat{\theta}_i$ such that

$$U_i(\theta_i) - U_i(\hat{\theta}_i) = \bar{s}(\lambda) \cdot (\theta_i - \hat{\theta}_i)$$

for some subgradient $\bar{s}(\lambda) \in \partial U_i(\theta_i(\lambda))$. This version of the Mean Value theorem (Clarke [2], p. 72) allows us to write, for any $\theta_i, \hat{\theta}_i$ in Θ_i :

$$U_i(\theta_i) = U_i(\hat{\theta}_i) + \int_0^1 \bar{s}(\lambda) \cdot (\theta_i - \hat{\theta}_i) \, d\lambda , \quad (22)$$

where for each $\lambda \in (0, 1)$, the vector $\bar{s}(\lambda)$ is any selection from the subdifferential $\partial U_i(\theta_i(\lambda))$. Thus, the proof will be completed if we show that any subgradient $s(\lambda)$ of $\nu_i(\theta_i(\lambda), \cdot)$ at $\theta_i(\lambda)$ is also a subgradient of the value function U_i at $\theta_i(\lambda)$.

Fix $\lambda \in (0, 1)$. Choose any subgradient $s(\lambda)$ of the function $\nu_i(\theta_i(\lambda), \cdot)$ at $\theta_i(\lambda)$, so that for any direction $y \in \mathbb{R}^{k_i}$ it holds that $D^+ \nu_i(\theta_i(\lambda), \theta_i(\lambda); y) \geq s(\lambda) \cdot y$ (where the one-sided directional derivative of ν_i is taken with respect to its second argument). The definition of

the value function U_i for the Bayesian Nash incentive compatible problem of agent i implies that, for any direction $y \in \mathbb{R}^{k_i}$, for $\delta > 0$,

$$\frac{\nu_i(\theta_i(\lambda), \theta_i(\lambda) + \delta y) - \nu_i(\theta_i(\lambda), \theta_i(\lambda))}{\delta} \leq \frac{U_i(\theta_i(\lambda) + \delta y) - U_i(\theta_i(\lambda))}{\delta}.$$

Taking limits in the above expression as δ approaches 0, one has that $D^+U_i(\theta_i(\lambda); y) \geq D^+\nu_i(\theta_i(\lambda), \theta_i(\lambda); y)$ for any direction y . It then follows readily that any vector $s(\lambda)$ in $\partial\nu_i(\theta_i(\lambda), \theta_i(\lambda))$ is also in $\partial U_i(\theta_i(\lambda))$. We can replace $\bar{s}(\lambda)$ with $s(\lambda)$ in expression (22) to obtain (19). ■

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