Efficient, ex post mechanism design: a differentiable approach*

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Abstract

This paper deals with efficient, ex post incentive compatible mechanism design in a social choice framework with quasi-linear utilities, interdependent valuations, and multi-dimensional types. We work with valuations that are linear in individual types and with a convex allocation set that lies in a multi-dimensional Euclidean space. A complete characterization of ex post incentive compatible allocation rules is provided, from which a simple necessary condition for ex post implementation is derived. Using a differential approach, we first show that in the one-dimensional case a surplus extraction tax scheme implements the efficient allocation rule ex post. We extend this insight and show by construction the existence of efficient, ex post incentive compatible mechanisms in two economically important classes of models with multi-dimensional types: in separable environments, i.e., when valuations for the allocation are separable, so that each dimension of the type space interacts with one dimension of the allocation; and in quasi-separable environments, where in addition several dimensions of the type space interact with some dimension of the social allocation in a specific way.

Keywords: interdependent valuations, ex post mechanisms, allocative efficiency, multi-dimensional types, public good provision.

JEL Classification Numbers: D71, D82, H41.

1 Introduction

Consider the fundamental problem of mechanism design theory, where a central planner (or a collective decision body) uses a direct revelation mechanism to elicit private information from agents in the economy to select the efficient allocation among a set of social alternatives. It is well known that if preferences for social alternatives depend only on each agent's information – the *private valuation* case – then the class of Vickrey-Clarke-Groves (VCG) mechanisms[1]

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[1]Vickrey (1961), Clarke (1971), and Groves (1973).

consists of dominant strategy incentive compatible mechanisms whose allocation rules select the efficient outcome. It is also known that VCG mechanisms do not work, in general, when preferences for social allocations are contingent on other agents' information as well – the *interdependent valuation* case. This is regrettable, since the private valuation assumption seems inadequate in many circumstances of economic interest. As an example, consider the allocation of public resources to finance improvements in an educational institution. In this application, the value of the selected alternative for an agent depends on information privately held by others: the quality of education a student receives may depend on how well motivated other students are, on their appreciation for science, math, arts, etc.

In recent years, two important papers have exposed the limitations of mechanism design with interdependent valuations in the presence of multi-dimensional private information. Jehiel and Moldovanu (2001) explore conditions under which efficient, Bayesian Nash incentive compatible mechanisms exist. They consider an economy with I agents, indexed by $i = 1, \ldots, I$, quasi-linear preferences that are, in addition, linear in types, and a finite set of alternatives, $k = 1, \ldots, K$. Each agent i receives an independent K-dimensional private signal $\theta^i = (\theta^i_1, \ldots, \theta^i_K)$. Valuations are interdependent in that the k-th component of θ^i – which we shall refer to as the k-th dimension of information – enters agent j's assessment of alternative k. In this framework, Jehiel and Moldovanu (2001) prove that efficient, Bayesian Nash direct mechanisms exist only if a congruence condition relating private and social rates of information substitution is satisfied. This condition is non-generic in that the set of parameters for which it is satisfied has measure zero.

In a sequel, Jehiel, Meyer-ter-Vehn, Moldovanu, and Zame (2006) present a stronger impossibility result that applies to ex post mechanisms. A direct mechanism is ex post incentive compatible if truth-telling by all agents constitutes an ex post equilibrium of the incomplete information game induced by the mechanism. Jehiel et al show that it is generally impossible to obtain non-trivial (i.e., non-constant) allocation rules that satisfy the ex post incentive constraints in a setting with 2 agents, 2 alternatives, interdependent, quasi-linear preferences and multi-dimensional types (not even without the allocative efficiency requirement). Jehiel, Meyer-ter-Vehn, Moldovanu, and Zame (2006) use the ex post taxation principle to express the incentive constraints in terms of a relationship between social and private indifference sets (subsets of the information space in which both alternatives are indifferent, from the social and individual perspective, respectively). They show that for any allocation rule to be ex post implementable, a geometric condition must be satisfied in these indifference sets, which are multi-dimensional submanifolds of the information space. The impossibility result comes from the fact that this condition is not satisfied by generic valuation functions.[2]

Interdependent valuations and multi-dimensional types are present in many economic situations of relevance, making the impossibility results in the literature very troublesome. On the other hand, several authors have argued in favor of ex post incentive compatibility as a very desirable property from the point of view of implementation, since it is robust to errors in the beliefs that agents hold about other agents characteristics.[3] The main purpose

[2]Jehiel and Moldovanu (2006) present a useful survey on this and related topics.

[3]See Bergemann and Morris (2005), Jehiel, Meyer-ter-Vehn, Moldovanu, and Zame (2006), and McLean and Postlewaite (2006), among others.

of this paper is to present some *positive* results for efficient, ex post mechanism design in a social choice setting with interdependent valuations and several dimensions of information. In our framework, which is formally presented in Subsection 2.1, agent i's type space is a K-dimensional closed interval, and his type θ^i is payoff relevant to all agents. Preferences for the social alternative and a transferable good (money) are quasi-linear; in addition, we assume that preferences are linear in individual types. Our setup considers a convex set of social alternatives that lies in a M-dimensional Euclidean space. A central planner uses a direct revelation mechanism to select, for each reported type profile, the allocation that maximizes the sum of valuations for the social alternative minus the cost of its provision, and the appropriate distribution of the tax burden to induce truthful revelation as an ex post equilibrium.

This framework encompasses many economic situations of interest; as an example, consider the classic public good provision problem. In this application, the social alternative $x = (x_1, \ldots, x_M)$ is a public good composed of M different projects of variable size (e.g., in an educational institution, x_1 is the size of science related facilities, x_2 is the size of art related facilities, x_3 is the size of the central library building, etc.). Agents in the economy estimate the value the public good according to more than one dimension (appreciation for science, appreciation for arts, etc.), and their estimates depend as well on information held by other agents. The central authority maximizes social welfare, interpreted here as the sum of valuations for the public good minus the cost of its provision. The central authority also designs a scheme to allocate the division of taxes among agents.

The linearity of valuations with respect to individual types is a convenient assumption that allows us to obtain, as a first step towards our positive results, a complete characterization of ex post implementable allocation rules. Although formulated in terms of subgradients of convex functions, our characterization result follows standard arguments (see Section 2.2). An important consequence is the ex post payoff equivalence principle: in equilibrium, payoffs are determined by the allocation rule alone. We make further use of this characterization to derive a simple necessary condition that any ex post implementable allocation rule must satisfied. This restriction, which amounts to the solvability of a system of differential equations that come from the multi-dimensional incentive compatibility constraints, is not vacuous, not even in our linear setup (we illustrate this point with an example). It follows that further restrictions on the social choice framework need to be in place to overcome the impossibility result of Jehiel, Meyer-ter-Vehn, Moldovanu, and Zame (2006).

Section 3 contains the main contribution of this paper. We show existence of efficient, ex post incentive compatible direct mechanisms in two economically important classes of environments that are generated by imposing additional assumptions on the cost of provision of the social alternative and on the heterogeneity of preferences. These environments are fairly standard and apply naturally to a variety of economic situations. Moreover, once the additional hypotheses are in place, it is readily verified that the efficient allocation rule satisfies the necessary condition for ex post implementation. We then use a differential approach, first employed by Laffont and Maskin (1980) in dominant strategy mechanisms with private valuations, to recover agent i's incentive taxes from a system of differential equations describing i's ex post incentive constraints.

We start the analysis in Subsection 3.1, where types are assumed to be one-dimensional (in this case the impossibility result of Jehiel et al does not apply). Agent i's taxes are found by integrating the differential equation describing i's first order incentive constraint; the resulting surplus extraction tax scheme implements the efficient allocation rule ex post. There is previous work on efficient, ex post mechanism design with interdependent valuations and one-dimensional types; most of this work focuses on auction settings.[4] Our social choice framework is closer to the one considered by Jehiel and Moldovanu (2001), and Bergemann and Välimäki (2002). In both papers, it is shown that a crossing condition is sufficient to obtain efficient, ex post mechanisms with tax schemes that follow the rationale of VCG transfers. We consider our findings in the one-dimensional case to be of interest for a couple of reasons. First, our incentive tax scheme resembles those employed in a screening problem; accordingly, the tax burden is designed to extract all of i's surplus at every realization of types, minus a term corresponding to i's informational rent that is in place to provide the right incentives for truth-telling. Second, since ex post payoffs are determined by the allocation rule alone, one infers that the surplus extraction tax scheme and any other appropriate tax scheme, including generalized VCG taxes, are equivalent up to a constant. Third, we find advantageous to work with the surplus extraction tax scheme since it is easily adapted to the multi-dimensional environments considered subsequently.

Subsection 3.2 contains the first class of multi-dimensional environments for which we show existence of efficient, ex post direct mechanisms. In a *separable environment*, the dimension the allocation set coincides with the dimension of the type spaces, valuation functions and the cost function are additively separable across projects, and each dimension of information affects the assessment of one project.[5] An important implication of these hypotheses is that the efficient allocation rule will also be separable: each of its component functions is contingent solely on the corresponding dimension of information. This feature is key to obtain a positive result in separable environments, where it is possible to transform a multi-dimensional incentive problem in to several, one-dimensional problems, whose solution is known from the one-dimensional case. More precisely, each differential equation describing the ex post incentive constraints of agent i in the k-th dimension of information ($k = 1, \ldots, K$) is independent of the others; it follows that the system of differential equations can be easily solved. Agent i's incentive tax is additively separable, and each of its components corresponds to a surplus extraction tax for the k dimension of the allocation. We illustrate our differential approach in separable environments with an example.

In Subsection 3.3 we broaden the class of environments to allow for different dimensions of preference heterogeneity to interact with some projects simultaneously; however, this form of interaction must be of a particular kind. In a *quasi-separable environment*, the dimension of the allocation set is greater than the dimension of the type spaces. For simplicity, we assume that the dimension of the allocation set is $K + 1$. The cost function in a quasi-separable environment is additively separable across projects. Moreover, the k-th dimension of information affects the assessment of projects k and $K + 1$; and this last interaction is

[4]For instance, Ausubel (2004), Dasgupta and Maskin (2000), and Perry and Reny (2002).

[5]These environments have been successfully used in other areas of information economics; see for example Rochet and Choné (1998), Manelli and Vincent (2006), and Fang and Norman (2006).

uniform across dimensions, i.e., marginal variations in the k-th and l-th dimensions of private information have the same impact on the assessment of project $K + 1$. An implication of the hypotheses made in the quasi-separable case is that the efficient allocation rule will be quasi-separable in the following sense: the first K component functions will be contingent solely on the corresponding dimension of information, and the last component function will depend on all dimensions of information uniformly. This property is sufficient to guarantee that the necessary condition for ex post implementation is satisfied; one can solve the system of differential equations describing the first order conditions of the incentive constraints to construct agent i's ex post incentive tax burden. Taxes paid by agent i reflect this structure. They are composed of $K + 1$ terms; each of the first K terms has a form already seen in the one-dimensional case and is aimed at solving the incentive problem related to the allocation of project k, the last term depends uniformly on all dimensions of information. The logic exposed before prevails: taxes paid by agent i are designed to extract all surplus generated by the efficient allocation, safe for the informational rent needed to ensure truth-telling. As before, we illustrate our approach at work in quasi-separable environments with an example.

To put our positive results in perspective with the impossibility result of Jehiel, Meyer-ter-Vehn, Moldovanu, and Zame (2006), we observe that it does not apply to separable environments, since the efficient allocation rule is itself separable. We argue, at the end of Section 3.3, that this powerful result is overcome in quasi-separable environments as well. Indeed, in such cases, the efficient allocation rule is quasi-separable and every dimension of information has equal marginal impact on the appraisal of project $K+1$. As as a consequence, the social indifference sets have dimension zero, and the geometric condition of Jehiel et al can be satisfied. Some concluding remarks are gathered in Section 4.

2 Preliminaries

2.1 The general setup

We consider a social choice model for an economy composed of I agents, indexed by $i = 1, \ldots, I$. A *social outcome* is a pair (x, t) that consists of a *social allocation* $x = (x_1, \ldots, x_M)$ belonging to the set $\mathcal{X}$, and a profile of *taxes* $t = (t^1, \ldots, t^I)$ belonging to $\mathbb{R}^I$. We assume that the allocation set $\mathcal{X}$ is equal to $\mathbb{R}^M_+$ (with $M \geq 1$). The cost of providing the allocation x is denoted by $C(x)$, where C is an increasing, continuously differentiable, real-valued function defined on $\mathcal{X}$. Each agent $i = 1, \ldots, I$ is endowed with a privately known, multi-dimensional type $\theta^i = (\theta^i_1, \ldots, \theta^i_K)$ that captures individual characteristics affecting i's preferences for the social allocation (more generally, it captures any privately held information pertinent to the valuation of x). Agent i's type space is $\Theta^i = \times_{k=1}^K [0, \overline{\theta}^i_k] \subseteq \mathbb{R}^K$ (with $K \leq M$). We shall refer to the k-th coordinate of θ^i as the k-th dimension of private information. If $K = 1$, then type $\theta^i = \theta^i_1$ captures all relevant information about individual characteristics necessary to estimate the value of the allocation x. If $K = M$, a common interpretation is that the assessment of project x_k, the k-th component of the social allocation, is contingent on the k-th dimension of private information. More generally, θ^i_k may interact with the entire allocation x. Let θ denote the profile of types of all agents, $\theta = (\theta^1, \ldots, \theta^I)$ and

$\Theta = \times_{i=1}^{I} \Theta^i$. As is standard practice, let θ^{-i} be the profile of types of all agents other than i, $\theta^{-i} = (\theta^1, \ldots, \theta^{i-1}, \theta^{i+1}, \ldots, \theta^I)$, and similarly $\Theta^{-i} = \times_{j \neq i} \Theta^j$. We shall write θ_k to denote the profile $(\theta_k^1, \ldots, \theta_k^I)$, and similarly θ_k^{-i}, for $k = 1, \ldots, K$.

We deal with preferences for social outcomes that are (i) quasi-linear with respect to taxes (money, the transferable good in this economy); (ii) linear in individual types, so that θ^i enters i's utility function in a linear way; and (iii) interdependent, so that agent j's type is payoff relevant to i. Thus, given a social outcome $(x, t) \in \mathcal{X} \times \mathbb{R}^I$ and a type profile $\theta \in \Theta$, agent i's utility is represented by

$$(1) \qquad \mathcal{U}^i\big(x, \theta, t^i\big) \;=\; \sum\nolimits_{k=1}^{K} \theta_k^i v_k^i\big(x, \theta^{-i}\big) \;-\; t^i \;=:\; V^i\big(x, \theta\big) \;-\; t^i.$$

Each term $\theta_k^i v_k^i\big(x, \theta^{-i}\big)$ in expression (1) captures the effect of the k-th dimension of private information held by agent i's on his assessment of the allocation x. Throughout the paper, we assume the following single crossing condition: for all $i = 1, \ldots, I$, $k = 1, \ldots, K$, and θ^{-i}, the appraisal function $v_k^i\big(\cdot, \theta^{-i}\big)$ is increasing and continuously differentiable on $\mathcal{X}$. The function $V^i : \mathcal{X} \times \Theta \to \mathbb{R}$ defined in (1) is called agent i's *valuation function.* Observe that V^i is indeed linear in i's types.[6]

We invoke the Revelation Principle and focus on direct revelation mechanisms. A central planner uses a direct mechanism Γ to elicit private information and select a social outcome (x, t). The mechanism Γ consists of an *allocation rule* ψ and a *tax scheme* τ: for every type profile reported to the central authority, ψ selects a social alternatives and τ specifies the division of the tax burden among the agents; i.e., $\Theta \ni \theta \to (\psi[\theta], \tau[\theta]) \in \mathcal{X} \times \mathbb{R}^I$.

The direct mechanism $\Gamma^* = (\psi^*, \tau)$ is said to be allocative *efficient* if its allocation rule ψ^* is efficient:

$$(2) \qquad \psi^*[\theta] \;\in\; \arg\max_{x \in \mathcal{X}} \Big\{ \sum\nolimits_{i=1}^{I} V^i\big(x, \theta\big) \;-\; C(x) \Big\}, \qquad \forall\, \theta \in \Theta\ .$$

The following is a standing hypothesis throughout the paper.

Assumption 1. *For every type profile θ in Θ, there exists a unique allocation $\psi^*[\theta] \in \mathcal{X}$ satisfying $\sum_{i=1}^{I} V^i\big(\psi^*[\theta], \theta\big) - C\big(\psi^*[\theta]\big) \;=\; \max_{x \in \mathcal{X}} \big\{ \sum_{i=1}^{I} V^i\big(x, \theta\big) - C(x) \big\}$. Moreover, the allocation rule $\psi^* : \Theta \to \mathcal{X}$ is continuously differentiable on Θ.*[7]

Assumption 1 guarantees the existence of an efficient, continuously differentiable allocation rule. Its implementation remains unresolved: agents in the economy do not necessarily share an interest in truthfully revealing their types. We employ ex post equilibrium as our solution concept. In an ex post equilibrium, i's strategy is optimal against the strategies of all other agents, for each realization of the other agents' types. The direct mechanism $\Gamma = (\psi, \tau)$

[6] Note also that bilinearity with respect to $(\theta^1, \ldots, \theta^I)$ is accommodate immediately. An alternative formulation would be to assume that i's valuation is linear in all types.

[7] Different premises on valuation functions and on the cost function yield to our Assumption 1. It would suffice to consider twice continuously differentiable functions and impose, say, strict concavity on the functions v_k^i over $\mathcal{X}$ and strict convexity on C over $\mathcal{X}$.

is said to be *ex post incentive compatible* if the tax scheme τ makes truth-telling by all agents an ex post equilibrium of the incomplete information game induced by Γ, in which case we say that ψ is *ex post implementable* by τ. That is, $\Gamma = (\psi, \tau)$ is ex post incentive compatible if for all $i = 1, \ldots, I$,

$$(3) \qquad \theta^i \;\in\; \arg\max_{\hat{\theta}^i \in \Theta^i} \Big\{ V^i\big(\psi[\hat{\theta}^i, \theta^{-i}], \theta\big) \;-\; \tau^i[\hat{\theta}^i, \theta^{-i}] \Big\}, \qquad \forall\, \theta \in \Theta.$$

Our framework accommodates several important economic applications, including the example mentioned in the introduction: $x = (x_1, \ldots, x_M)$ is a public good composed of M different projects of variable size; agents in the economy estimate the value the public good according to several dimensions, and their assessments depend as well on information held by others. The central authority maximizes social welfare, interpreted here as the sum of valuations for the public good minus the cost of its provision. In this particular application it would be natural to ask whether an ex post incentive compatible mechanism $\Gamma = (\psi, \tau)$ is also ex post feasible, in the sense that for each reported type profile θ in Θ, the sum of taxes covers the cost of provision of the public good: $\sum_{i=1}^{I} \tau^i[\theta] \;\geq\; C\big(\psi[\theta]\big)$.[8]

2.2 Characterization of ex post implementable allocation rules

Suppose the central planner uses the direct revelation mechanism $\Gamma = (\psi, \tau)$ to elicit private information and select a social outcome (x, t) in $\mathcal{X} \times \mathbb{R}^I$. Fix a type profile $\theta^{-i} \in \Theta^{-i}$. If all agents but i are truthfully announcing their types θ^{-i}, let

$$\overline{U}^i\big(\hat{\theta}^i, \theta^i; \theta^{-i}, \Gamma\big) \;\;:=\;\; V^i\big(\psi[\hat{\theta}^i, \theta^{-i}], \theta\big) \;-\; \tau^i[\hat{\theta}^i, \theta^{-i}]$$

denote the utility of agent i when his type is θ^i and his announcement is $\hat{\theta}^i$. We can write agent i's ex post incentive compatibility requirement of expression (3) as follows:

$$(4) \qquad U^i\big(\theta^i; \theta^{-i}, \Gamma\big) \;:=\; \overline{U}^i\big(\theta^i, \theta^i; \theta^{-i}, \Gamma\big) \;\geq\; \overline{U}^i\big(\hat{\theta}^i, \theta^i; \theta^{-i}, \Gamma\big), \qquad \forall\, \theta^i, \hat{\theta}^i \in \Theta^i.$$

The indirect utility function $U^i\big(\cdot; \theta^{-i}, \Gamma\big)$ defined above is the value function of the incentive problem of agent i when truth-telling is an optimal strategy ex post. The following technical observation regarding its properties will be needed.

Lemma 1. *If the direct revelation mechanism $\Gamma = (\psi, \tau)$ is ex post incentive compatible, then for each agent $i = 1, \ldots, I$, and for each profile of types θ^{-i} in Θ^{-i}, the indirect utility function $\theta^i \to U^i\big(\theta^i; \theta^{-i}, \Gamma\big)$ defined in expression* (4) *is convex on Θ^i. Moreover, if θ^i and $\hat{\theta}^i$ belong to Θ^i then*

$$(5) \qquad U^i\big(\theta^i; \theta^{-i}, \Gamma\big) \;=\; U^i\big(\hat{\theta}^i; \theta^{-i}, \Gamma\big) \;+\; \int_0^1 \partial U^i\big(\theta^i(\lambda); \theta^{-i}, \Gamma\big) \cdot \big(\theta^i - \hat{\theta}^i\big)\, \mathrm{d}\lambda;$$

[8]Ideally, Γ would be ex post budget balanced, so that for every realization of types, the sum of taxes equals the cost of providing the allocation. This condition is very demanding. Even in private valuation settings where efficient, dominant strategy implementation is obtained by means of VCG mechanisms, budget balancedness is generally not satisfied (Green and Laffont (1979), Laffont and Maskin (1980)).

where for all $\lambda \in (0,1)$, $\theta^i(\lambda) := \lambda\theta^i + (1-\lambda)\hat{\theta}^i$, and $\partial U^i\big(\theta^i(\lambda);\theta^{-i},\Gamma\big)$ is the subdifferential of the indirect utility function $U^i\big(\cdot;\theta^{-i},\Gamma\big)$ evaluated at $\theta^i(\lambda)$.

The meaning of equation (5) is the following.[9] Fix θ^i and $\hat{\theta}^i$ in $\Theta^i \subseteq \mathbb{R}^K$; for every $0 < \lambda < 1$, let $\theta^i(\lambda)$ be a point in the line segment connecting θ^i and $\hat{\theta}^i$ and $s(\lambda)$ be any selection of the subdifferential set $\partial U^i\big(\theta^i(\lambda);\theta^{-i},\Gamma\big) \subseteq \mathbb{R}^K$. Then the value of the integral $\int_0^1 s(\lambda)\cdot\big(\theta^i - \hat{\theta}^i\big)\,\mathrm{d}\lambda$ is equal to $U^i\big(\theta^i;\theta^{-i},\Gamma\big) - U^i\big(\hat{\theta}^i;\theta^{-i},\Gamma\big)$, and is independent of the selection that one uses.

Proof Fix a profile θ^{-i} in Θ^{-i}. It is clear that for any report $\hat{\theta}^i$, the function $\theta^i \to \overline{U}^i\big(\hat{\theta}^i,\theta^i;\theta^{-i},\Gamma\big)$ is linear. Indeed, using (1),

$$\begin{aligned}\overline{U}^i\big(\hat{\theta}^i,\theta^i;\theta^{-i},\Gamma\big) &= V^i\big(\psi[\hat{\theta}^i,\theta^{-i}],\theta^i,\theta^{-i}\big) - \tau^i[\hat{\theta}^i,\theta^{-i}] \\ &= \sum_{k=1}^K \theta^i_k v^i_k\big(\psi[\hat{\theta}^i,\theta^{-i}],\theta^{-i}\big) - \tau^i[\hat{\theta}^i,\theta^{-i}].\end{aligned}$$

Convexity of $U^i\big(\cdot;\theta^{-i},\Gamma\big)$ follows from the well-known result that the value function of a family of convex functions is itself convex. Expression (5) is the integral form of the mean value theorem for subdifferentials of convex functions (see Hiriart-Urruty and Lemaréchal (2001), Theorem 2.3.4, Chapter D). ■

We use Lemma 1 to establish a complete characterization of ex post implementable allocation rules when valuations are interdependent and linear in their own types. To economize on notation, given a profile of types θ^{-i} in Θ^{-i} and an allocation rule ψ, define agent i's *reduced form allocation*[10] as the vector-valued function $\mathbf{v}^i\big(\cdot;\theta^{-i},\psi\big)$ defined on Θ^i by

$$(6)\quad \mathbf{v}^i\big(\theta^i;\theta^{-i},\psi\big) = \big(\,\mathbf{v}^i_k\big(\theta^i;\theta^{-i},\psi\big)\,\big)_{k=1}^K := \big(\,v^i_1\big(\psi[\theta],\theta^{-i}\big),\ \ldots\ ,v^i_K\big(\psi[\theta],\theta^{-i}\big)\,\big)\,.$$

When all agents are truthfully reporting, the vector $\mathbf{v}^i\big(\theta^i;\theta^{-i},\psi\big)$ encompasses type θ^i's value of the allocation selected under ψ. The reduced form allocation also captures the direct

[9] We recall that the one-sided directional derivative of a convex function $f:\mathbb{R}^p\to\mathbb{R}$, evaluated at $y\in\mathbb{R}^p$ in the direction $d\in\mathbb{R}^p$, is defined by:

$$D^+f(y;d) := \lim_{\lambda\downarrow 0}\frac{f(y+\lambda d)-f(y)}{\lambda},$$

where, by the convexity of f, this limit exists for all y and all d. The subdifferential $\partial f(y)$ of the function f at the point y is the non-empty, convex and compact subset of $\mathbb{R}^p$ defined by

$$\partial f(y) := \{s\in\mathbb{R}^p \mid s\cdot d \le D^+f(y;d),\quad \forall\, d\in\mathbb{R}^p\}.$$

A vector s in $\partial f(y)$ is called a subgradient of f at y. Observe that f is differentiable at y if and only if its unique subgradient at y is the gradient vector $\nabla f(y)$. Related results about convex functions and subdifferentials can be found in Hiriart-Urruty and Lemaréchal (2001).

[10] The term is borrowed from Ledyard and Palfrey (2007), who use it in a social choice model with one-dimensional types and linear, private value preferences.

impact, in terms of i's preferences, of a marginal change in θ^i when all other agents are truth-telling.[11] Suppose that the allocation rule ψ is ex post implementable by some tax scheme τ, and write $\Gamma = (\psi, \tau)$. If the indirect utility function $U^i\big(\cdot;\theta^{-i},\Gamma\big)$ is differentiable at θ^i, then one readily obtains using the envelope theorem the following relation:

$$\begin{aligned}\nabla U^i\big(\theta^i;\theta^{-i},\Gamma\big) &= \left.\frac{\partial\big\{V^i\big(\psi[\tilde\theta^i,\theta^{-i}],\theta^i,\theta^{-i}\big) \;-\; \tau^i[\tilde\theta^i,\theta^{-i}]\big\}}{\partial\theta^i}\right|_{\tilde\theta^i=\theta^i}\\ &= \Big(\, v_1^i\big(\psi[\tilde\theta^i,\theta^{-i}],\theta^{-i}\big),\;\ldots\;, v_K^i\big(\psi[\tilde\theta^i,\theta^{-i}],\theta^{-i}\big)\,\Big)\Big|_{\tilde\theta^i=\theta^i}\\ &= \mathbf{v}^i\big(\theta^i;\theta^{-i},\psi\big).\end{aligned}$$

If $U^i\big(\cdot;\theta^{-i},\Gamma\big)$ is not differentiable at θ^i (an event that happens almost nowhere in Θ^i, since the indirect utility function $U^i\big(\cdot;\theta^{-i},\Gamma\big)$ is convex and therefore differentiable a.e. on Θ^i), one can nonetheless show that the vector $\mathbf{v}^i\big(\theta^i;\theta^{-i},\psi\big)$ is a subgradient of the indirect utility $U^i\big(\theta^i;\theta^{-i},\Gamma\big)$.[12] It follows that the value of the reduced form allocation $\mathbf{v}^i\big(\cdot;\theta^{-i},\psi\big)$ is in the subdifferential set $U^i\big(\theta^i;\theta^{-i},\Gamma\big)$, for every type θ^i in Θ^i. Moreover, since the indirect utility of i is a convex function, the reduced form allocation generated by ψ is monotone (in the sense of condition (ii) of Proposition 1). This observation leads us to the following characterization of ex post implementable allocation rules in terms of reduced form allocations.

Proposition 1. *The allocation rule $\psi:\Theta\to\mathcal{X}$ is ex post implementable by some transfer scheme $\tau:\Theta\to\mathbb{R}^I$, in which case we say that the mechanism $\Gamma=(\psi,\tau)$ is ex post incentive compatible, if and only if the following two conditions hold:*

(i) For every agent $i=1,\ldots,I$, for every profile θ^{-i} in Θ^{-i}, for any two θ^i and $\hat\theta^i$ belonging to Θ^i, it is the case that

$$U^i\big(\theta^i;\theta^{-i},\Gamma\big) \;=\; U^i\big(\hat\theta^i;\theta^{-i},\Gamma\big) \;+\; \int_0^1 \mathbf{v}^i\big(\theta^i(\lambda);\theta^{-i},\psi\big)\cdot(\theta^i-\hat\theta^i)\,\mathrm{d}\lambda;$$

where $\lambda\in(0,1)$ and $\theta^i(\lambda):=\lambda\theta^i+(1-\lambda)\hat\theta^i$.

(ii) For every agent $i=1,\ldots,I$, for every profile θ^{-i} in Θ^{-i}, the reduced form allocation $\mathbf{v}^i\big(\cdot;\theta^{-i},\psi\big)$ defined in (6) is monotone; i.e.,

$$\big\{\mathbf{v}^i\big(\theta^i;\theta^{-i},\psi\big) \;-\; \mathbf{v}^i\big(\hat\theta^i;\theta^{-i},\psi\big)\big\}\cdot\big\{\theta^i \;-\; \hat\theta^i\big\} \;\geq\; 0, \qquad \forall\ \theta^i,\hat\theta^i\in\Theta^i.$$

Proof See appendix. ■

Although conveniently formulated in terms of subgradients of convex functions, the logic of this characterization follows standard arguments (e.g., the classical paper of Myerson (1981)).

[11] Jehiel and Moldovanu (2001) call it the individual rate of information substitution.

[12] See Hiriart-Urruty and Lemaréchal (2001), Theorem 4.4.2, Chapter D.

The incentive compatibility constraints are replaced by first and second order conditions, which are sufficient as well as necessary when valuations are linear in individual types.[13] An immediate consequence of Proposition 1 is a payoff equivalence principle for ex post incentive compatible mechanisms: for every realization of types θ^{-i}, agent i's payoff is determined by the allocation rule alone. Thus, given any two ex post incentive compatible mechanisms Γ and Γ' that share the allocation rule ψ, and given any realization θ^{-i}, payoffs to type θ^i generated by Γ and Γ' are equal up to an additive constant.

One shall not conclude that the conditions of Proposition 1 are generally satisfied. Jehiel, Meyer-ter-Vehn, Moldovanu, and Zame (2006) have shown that in environments with several dimensions of information, the ex post implementation of non-constant allocation rules is non-generic. Even in our linear framework, the restrictions imposed by the multi-dimensional incentive compatibility constraints are stringent: they translate into an integrability condition that is not generally satisfied. We derive the following version of the aforementioned integrability condition that. Suppose that the allocation rule ψ is continuously differentiable and ex post implementable by transfers τ, and put $\Gamma = (\psi, \tau)$. It follows that the indirect utility function $U^i\big(\cdot;\theta^{-i},\Gamma\big)$ is convex on Θ^i. If $K = 1$ (and thus Θ^i is an interval), the convexity of the indirect utility function implies that it is twice differentiable almost everywhere. This remarkable result extends to higher dimensions.[14] Thus, $U^i\big(\cdot;\theta^{-i},\Gamma\big)$ is twice differentiable almost everywhere on $\Theta^i \subseteq \mathbb{R}^K$, for $K \geq 1$. Moreover, if θ^i is a point where the second derivative exists, then the Hessian matrix $\mathrm{D}\big[\nabla U^i\big(\theta^i;\theta^{-i},\Gamma\big)\big] = \mathrm{D}\big[\mathbf{v}^i\big(\theta^i;\theta^{-i},\psi\big)\big]$ is symmetric and positive semi-definite. Using (6), it must be that at any such point of differentiation, and for any two dimensions of private information $k, l = 1, \ldots, K$:

$$\frac{\partial^2 U^i\big(\theta^i;\theta^{-i},\Gamma\big)}{\partial\theta^i_l\,\partial\theta^i_k} = \frac{\partial}{\partial\theta^i_l}\Big\{v^i_k\big(\psi[\theta^i,\theta^{-i}],\theta^{-i}\big)\Big\} = \frac{\partial v^i_k}{\partial x}\big(\psi[\theta^i,\theta^{-i}],\theta^{-i}\big)\cdot\frac{\partial\psi}{\partial\theta^i_l}[\theta^i,\theta^{-i}];$$

and similarly

$$\frac{\partial^2 U^i\big(\theta^i;\theta^{-i},\Gamma\big)}{\partial\theta^i_k\,\partial\theta^i_l} = \frac{\partial}{\partial\theta^i_k}\Big\{v^i_l\big(\psi[\theta^i,\theta^{-i}],\theta^{-i}\big)\Big\} = \frac{\partial v^i_l}{\partial x}\big(\psi[\theta^i,\theta^{-i}],\theta^{-i}\big)\cdot\frac{\partial\psi}{\partial\theta^i_k}[\theta^i,\theta^{-i}].$$

The symmetry of $\mathrm{D}\big[\nabla U^i\big(\theta^i;\theta^{-i},\Gamma\big)\big]$ now implies that if the allocation rule ψ is ex post implementable, the following restriction must be satisfied for all $k, l = 1, \ldots, K$:

$$\frac{\partial v^i_k}{\partial x}\big(\psi[\theta^i,\theta^{-i}],\theta^{-i}\big)\cdot\frac{\partial\psi}{\partial\theta^i_l}[\theta^i,\theta^{-i}] = \frac{\partial v^i_l}{\partial x}\big(\psi[\theta^i,\theta^{-i}],\theta^{-i}\big)\cdot\frac{\partial\psi}{\partial\theta^i_k}[\theta^i,\theta^{-i}]. \tag{7}$$

We summarize this as follows.

[13] A similar characterization for Bayesian Nash allocation rules has been presented by several authors, among others Jehiel and Moldovanu (2001), Ledyard and Palfrey (2007), Rochet (1987), and Rochet and Choné (1998); for a survey, see Rochet and Stole (2003). Krishna and Maenner (2001) explore the use of convex potentials in mechanism design.

[14] Its multi-dimensional version is known as Alexandrov's theorem; see Howard (1998).

Corollary 1. *The continuously differentiable allocation rule* ψ *is ex post implementable only if for every agent* $i = 1, \ldots, I$, *for every type profile* θ^{-i} *in* Θ^{-i}, *and for any two dimensions of private information* $k, l = 1, \ldots, K$, *expression* (7) *is satisfied a.e. on* Θ^i.

One could expand the inner product operator and write condition (7) as:

$$\sum_{m=1}^{M} \left\{ \frac{\partial v_k^i\big(\psi[\theta], \theta^{-i}\big)}{\partial x_m} \frac{\partial \psi_m[\theta]}{\partial \theta_l^i} \right\} \quad = \quad \sum_{m=1}^{M} \left\{ \frac{\partial v_l^i\big(\psi[\theta], \theta^{-i}\big)}{\partial x_m} \frac{\partial \psi_m[\theta]}{\partial \theta_k^i} \right\}.$$

The above expression has the following interpretation. Suppose all agents are truthfully announcing their types. A variation in θ^i directly affects agent i's indirect utility function through the reduced form allocation $\mathbf{v}^i\big(\theta^i; \theta^{-i}, \psi\big)$. An additional change in one dimension of i's type, say θ_l^i, affects i's indirect utility through a marginal change in the reduced form allocation. The left hand side of the above equation captures the effect of a variation in θ_l^i over the k-th component of the vector of reduced form allocation. For ψ to be ex post implementable, this effect must be equal to the one produced on the l-th component of $\mathbf{v}^i\big(\theta^i; \theta^{-i}, \psi\big)$ when the θ_k^i is modified. Otherwise, agent i might find it profitable to deviate from truth-telling, say by increasing his report in one dimension while decreasing it in the other dimension.

If we now ask whether the efficient allocation rule ψ^* is ex post implementable, then by Corollary 1 expression (7) must be satisfied, for all $i = 1, \ldots, I$, when the reduced form allocation is generated by ψ^*. But ψ^* depends on all valuations and on the cost function, and thus in general expression (7) will be not satisfied for every agent in the economy.[15] It follows that the restrictions embodied in Corollary 1 limit the classes of models one can work with. We want to emphasize that these restrictions are originated in the interaction of preferences, costs, and the several dimensions of type heterogeneity. The next example serves as a simple illustration of the restrictions imposed by expression (7) when applied to efficient allocation rules.

Example 1. The set of agents in the economy is $I = \{1, 2, 3\}$. To simplify calculations, we assume that only agent 1 creates informational externalities, but this is irrelevant. The allocation set is $\mathcal{X} = \mathbb{R}_+^3$, so $M = 3$ and $x = (x_1, x_2, x_3)$. The type space of agent $j \in I$ is $\Theta^j = [0, \overline{\theta}_1^j] \times [0, \overline{\theta}_2^j]$, so $K = 2$. Valuations for social alternatives are given by:

$$\begin{aligned} V^1\big(x, \theta^1\big) \quad &= \quad \theta_1^1\big(\alpha_1^1 x_1 + \beta_1^1 x_2 + \gamma_1^1 x_3\big) \ + \ \theta_2^1\big(\alpha_2^1 x_1 + \beta_2^1 x_2 + \gamma_2^1 x_3\big); \\ V^i\big(x, \theta^1, \theta^i\big) \quad &= \quad \theta_1^i\big\{\theta_1^1\big(\alpha_1^i x_1 + \beta_1^i x_2 + \gamma_1^i x_3\big)\big\} \ + \ \theta_2^i\big\{\theta_2^1\big(\alpha_2^i x_1 + \beta_2^i x_2 + \gamma_2^i x_3\big)\big\}, \qquad i = 2, 3; \end{aligned}$$

where α_k^j, β_k^j and γ_k^j are known, positive parameters, for $j \in I$ and $k = 1, 2$. The cost of providing x is represented by $C(x) = \frac{1}{2}\sum_{m=1}^3 x_m^2$. Given reports θ, the central planner chooses the allocation that maximizes social welfare $\sum_j V^j(x, \theta) \ - \ C(x)$.

[15] Jehiel and Moldovanu (2004) use a similar intuition to underline the limitations of the design of private industries in the Bayesian Nash case; see also Jehiel and Moldovanu (2001) and Jehiel, Meyer-ter-Vehn, Moldovanu, and Zame (2006).

The efficient allocation rule $\theta \to \psi^*[\theta] = \big(\psi^*_1[\theta], \psi^*_2[\theta], \psi^*_3[\theta]\big)$ takes the form

$$\psi^*_1[\theta] \;=\; \theta^1_1\big(\alpha^1_1 + \alpha^2_1\theta^2_1 + \alpha^3_1\theta^3_1\big) \;+\; \theta^1_2\big(\alpha^1_2 + \alpha^2_2\theta^2_2 + \alpha^3_2\theta^3_2\big), \tag{8}$$

$$\psi^*_2[\theta] \;=\; \theta^1_1\big(\beta^1_1 + \beta^2_1\theta^2_1 + \beta^3_1\theta^3_1\big) \;+\; \theta^1_2\big(\beta^1_2 + \beta^2_2\theta^2_2 + \beta^3_2\theta^3_2\big), \tag{9}$$

$$\psi^*_3[\theta] \;=\; \theta^1_1\big(\gamma^1_1 + \gamma^2_1\theta^2_1 + \gamma^3_1\theta^3_1\big) \;+\; \theta^1_2\big(\gamma^1_2 + \gamma^2_2\theta^2_2 + \gamma^3_2\theta^3_2\big). \tag{10}$$

One shall now verify if ψ^* can be ex post implemented by some tax scheme. In particular, the following expression should hold for all type profiles θ:

$$\frac{\partial v^1_1}{\partial x}\big(\psi^*[\theta]\big) \cdot \frac{\partial \psi^*}{\partial \theta^1_2}[\theta] \;=\; \frac{\partial v^1_2}{\partial x}\big(\psi^*[\theta]\big) \cdot \frac{\partial \psi^*}{\partial \theta^1_1}[\theta].$$

In this example, the above requirement is expressed by the equality

$$\begin{aligned} \alpha^1_1\big(\alpha^2_2\theta^2_2 + \alpha^3_2\theta^3_2\big) \;+\; \beta^1_1\big(\beta^2_2\theta^2_2 + \beta^3_2\theta^3_2\big) \;+\; \gamma^1_1\big(\gamma^2_2\theta^2_2 + \gamma^3_2\theta^3_2\big) & \\ = \alpha^1_2\big(\alpha^2_1\theta^2_1 + \alpha^3_1\theta^3_1\big) \;+\; \beta^1_2\big(\beta^2_1\theta^2_1 + \beta^3_1\theta^3_1\big) \;+\; \gamma^1_2\big(\gamma^2_1\theta^2_1 + \gamma^3_1\theta^3_1\big), & \end{aligned}$$

that should hold for all θ in Θ. The set of parameters for which this is satisfied is obviously non-generic. □

3 Efficient, ex post incentive compatible mechanisms

This section explores the existence of efficient, ex post direct revelation mechanisms in two environments that encompass important economic applications, e.g., the provision of multi-dimensional public goods. Our approach is as follows. We impose additional restrictions to the general setup developed in Section 2, and show that different assumptions imply certain general properties of the resulting efficient allocation rule. As a second step, we verify that the necessary condition for ex post implementation is satisfied. Once the answer to this question is in the positive, we construct a tax scheme to implement the corresponding efficient allocation rule using the differentiable approach of Laffont and Maskin (1980). The existence of such tax scheme attests the solvability of a certain system of differential equations that is congruent with condition (7). To help gain intuition, we start with the one-dimensional case. We will show in Subsection 3.2 that if the social environment satisfies a separability condition, the positive result of the one-dimensional case extends readily to the multi-dimensional case. A broader class of environments is considered in Subsection 3.3.

3.1 One-dimensional environments

If types are one-dimensional ($K = 1$), we abuse notation and write $\theta^i_1 = \theta^i$, $\Theta^i = [0, \overline{\theta}^i]$, and $v^i_1 = v^i$, for all $i = 1, \ldots, I$. For simplicity, we also assume that the allocation set is one-dimensional: $\mathcal{X} = \mathbb{R}_+$. Extensions to allocation sets of higher dimension are not difficult. Given a social outcome $(x,t) \in \mathcal{X} \times \mathbb{R}^I$, agent i's payoff takes now the form

$$\mathcal{U}^i(x, \theta, t^i) \;=\; \theta^i v^i(x, \theta^{-i}) \;-\; t^i;$$

where, as in the general case, it is assumed that for all i and all θ^{-i}, the function $v^i(\cdot,\theta^{-i})$ is increasing and continuously differentiable on $\mathcal{X}$. Similarly, the cost function C is now defined on $\mathbb{R}_+$. We maintain Assumption 1: there exists a continuously differentiable allocation rule $\psi^*:\Theta\to\mathcal{X}$ satisfying

$$\psi^*[\theta] \;=\; \arg\max_{x\geq 0}\Big\{\sum\nolimits_{i=1}^{I}\theta^i v^i(x,\theta^{-i}) \;-\; C(x)\Big\}, \qquad\qquad \forall\,\theta\in\Theta.$$

In the one-dimensional case, a direct revelation mechanism $\Gamma=(\psi,\tau)$ is ex post incentive compatible if for all i and all $\theta^{-i}\in\Theta^{-i}$, it is the case that

$$U^i\big(\theta^i;\theta^{-i},\Gamma\big) \;:=\; \overline{U}^i\big(\theta^i,\theta^i;\theta^{-i},\Gamma\big) \;\geq\; \overline{U}^i\big(\hat{\theta}^i,\theta^i;\theta^{-i},\Gamma\big), \qquad \forall\,\theta^i,\hat{\theta}^i\in[0,\overline{\theta}^i];$$

where, abusing notation, we let $\overline{U}^i\big(\hat{\theta}^i,\theta^i;\theta^{-i},\Gamma\big) \;:=\; \theta^i v^i\big(\psi[\hat{\theta}^i,\theta^{-i}],\theta^{-i}\big)-\tau^i[\hat{\theta}^i,\theta^{-i}]$. Note also that given ψ and θ^{-i}, agent i's reduced form allocation generated by ψ is now a real-valued function denoted by $\theta^i\to\mathbf{v}^i\big(\theta^i;\theta^{-i},\psi\big)=v^i\big(\psi[\theta^i,\theta^{-i}],\theta^{-i}\big)$. The characterization result of Proposition 1 should be stated in terms of these reduced form allocations. We make the observation that, in the one-dimensional case, the monotonicity requirement of condition (ii) in Proposition 1 is sharper: to be ex post implementable, the allocation rule ψ must be increasing.[16] Observe as well that the necessary condition of Corollary 1 is trivially satisfied when types are unidimensional.

The differential approach of Laffont and Maskin (1980) is used to construct an efficient, ex post incentive compatible mechanism in the one-dimensional case. To that purpose, let ψ^* be the efficient, continuously differentiable allocation rule which, by condition (ii) in Proposition 1, must be increasing. We replace condition (i) of Proposition 1 with the construction of the appropriate tax scheme τ, so the ex post incentive compatibility constraints of every agent are satisfied. For that, the following relation is a necessary condition:

$$\theta^i\frac{\partial v^i}{\partial x}\big(\psi^*[\theta],\theta^{-i}\big)\frac{\partial\psi^*}{\partial\theta^i}[\theta^i,\theta^{-i}] \;-\; \frac{\partial\tau^i}{\partial\theta^i}[\theta^i,\theta^{-i}] \;=\; 0, \qquad \forall\,\theta\in\Theta.$$

Add and subtract $v^i\big(\psi^*[\theta],\theta^{-i}\big)$, and rearrange the above equation to obtain that taxes to agent i are described by the following differential equation:

$$\frac{\partial\tau^i}{\partial\theta^i}[\theta] \;=\; \theta^i\frac{\partial v^i}{\partial x}\big(\psi^*[\theta],\theta^{-i}\big)\frac{\partial\psi^*}{\partial\theta^i}[\theta] \;+\; v^i\big(\psi^*[\theta],\theta^{-i}\big) \;-\; v^i\big(\psi^*[\theta],\theta^{-i}\big), \qquad \forall\,\theta\in\Theta.$$

We can now integrate the above expression and use the definition of the reduced form allocation to find that agent i's tax schedule $\tau^i:\Theta\to\mathbb{R}$ takes the form

$$\tau^i[\theta] \;=\; \theta^i v^i\big(\psi^*[\theta],\theta^{-i}\big) \;-\; \int_0^{\theta^i}\mathbf{v}^i\big(\tilde{\theta}^i;\theta^{-i},\psi^*\big)\,\mathrm{d}\tilde{\theta}^i \;+\; h^i(\theta^{-i}). \tag{11}$$

[16] To see this, fix θ^{-i} and let $\hat{\theta}^i<\theta^i$. Using condition (ii) of Proposition 1, one has that $\mathbf{v}^i\big(\hat{\theta}^i;\theta^{-i},\psi\big)\leq\mathbf{v}^i\big(\theta^i;\theta^{-i},\psi\big)$. Since $v^i(x,\theta^{-i})$ is increasing in x for all θ^{-i}, it follows that $\psi[\hat{\theta}^i,\theta^{-i}]\leq\psi[\theta^i,\theta^{-i}]$.

Notice the logic behind the construction of the above incentive tax scheme. It is design to extract all of agent i's surplus generated under ψ^*, except for a informational rent that is left to ensure truthful revelation. This surplus extraction tax is derived using local necessary conditions. It remains to check that the direct mechanism $\Gamma = (\psi^*, \tau)$, where for all i, τ^i is defined in (11), is indeed efficient, ex post incentive compatible, i.e., satisfies all ex post incentive compatibility constraints. We do so in the next proposition.

Proposition 2. *In a one-dimensional environment, let ψ^* be the efficient, continuously differentiable allocation rule such that, for all $i = 1, \ldots, I$ and all $\theta^{-i} \in \Theta^{-i}$, the reduced form allocation $\mathbf{v}^i(\cdot; \theta^{-i}, \psi^*)$ is increasing in θ^i. Then ψ^* is ex post implementable by the tax scheme $\tau = (\tau^1, \ldots, \tau^I)$ if and only if for $i = 1, \ldots, I$, and for every profile θ in Θ, $\tau^i[\theta]$ is given by expression* (11).

Proof Let $\Gamma^* = (\psi^*, \tau)$. We have argued above that if ψ^* is ex post implementable by τ then expression (11) is satisfied. For the converse, it suffices to show that for every $i = 1, \ldots, I$, for every profile θ^{-i}, and any two $\theta^i, \hat{\theta}^i$ in Θ^i,

$$U^i(\theta^i; \theta^{-i}, \Gamma^*) \; - \; \overline{U}^i(\hat{\theta}^i, \theta^i; \theta^{-i}, \Gamma^*) \quad \geq \quad 0.$$

We use taxes τ^i as defined in (11) and the reduced form allocation $\mathbf{v}^i$ to express the above incentive compatibility constraint as:

$$\begin{aligned} & \Big\{\theta^i v^i\big(\psi^*[\theta], \theta^{-i}\big) \; - \; \tau^i[\theta]\Big\} \; - \; \Big\{\theta^i v^i\big(\psi^*[\hat{\theta}^i, \theta^{-i}], \theta^{-i}\big) \; - \; \tau^i[\hat{\theta}^i, \theta^{-i}]\Big\} \\ = \; & \int_{\hat{\theta}^i}^{\theta^i} \mathbf{v}^i\big(\tilde{\theta}^i; \theta^{-i}, \psi^*\big)\, \mathrm{d}\tilde{\theta}^i \; - \; \big(\theta^i - \hat{\theta}^i\big) v^i\big(\psi^*[\hat{\theta}^i, \theta^{-i}], \theta^{-i}\big) \\ = \; & \int_{\hat{\theta}^i}^{\theta^i} \Big\{\mathbf{v}^i\big(\tilde{\theta}^i; \theta^{-i}, \psi^*\big) \; - \; \mathbf{v}^i\big(\hat{\theta}^i; \theta^{-i}, \psi^*\big)\Big\}\, \mathrm{d}\tilde{\theta}^i \quad \geq \quad 0; \end{aligned}$$

where the last inequality follows from the monotonicity of the reduced form allocation $\mathbf{v}^i(\cdot; \theta^{-i}, \psi^*)$. Thus, the tax scheme τ implements the efficient allocation rule ψ^* ex post, as desired. ■

There is some previous work on efficient, ex post mechanisms with interdependent valuations and one-dimensional types, although most of the literature focuses on auction settings.[17] Our social choice framework is closer to the one considered by Jehiel and Moldovanu (2001), who deal with valuation functions that are linear in types and a finite set of allocations, and Bergemann and Välimäki (2002), who consider more general valuation functions and extend the analysis to a continuum of allocations. In both cases, it is shown that a monotonicity condition on the efficient allocation rule together with the standard single crossing condition suffice to achieve ex post implementation. Moreover, this is done by means of a transfer scheme whose logic resembles that of the VCG scheme – which, we shall recall, forces agent

[17] See Ausubel (2004), Dasgupta and Maskin (2000), Perry and Reny (2002).

i to take into account the social opportunity cost of his announcement. In contrast, we have derived the surplus extraction tax scheme τ directly using the differential approach. Its rationale can be understood if we compare our social choice problem with a screening problem. Accordingly, the tax burden of agent i is designed to extract all of i's ex post surplus, minus a term corresponding to i's informational rent. Note that Proposition 2 makes clear that the generalized VCG tax scheme, or any other incentive tax scheme that implements the efficient rule ex post, is equivalent to the surplus extraction tax scheme, up to an additive constant. Agent i's indirect utility, for each realization of types θ, is given by

$$U^i\big(\theta^i;\theta^{-i},\Gamma^*\big) \;=\; \int_0^{\theta^i} \mathbf{v}^i\big(\tilde{\theta}^i;\theta^{-i},\psi^*\big)\,\mathrm{d}\tilde{\theta}^i \;-\; h^i(\theta^{-i}).$$

Questions regarding other desirable properties of efficient, ex post incentive compatible mechanisms, such as feasibility or individual rationality, can now be addressed by exploring the existence of a profile $h = (h^1,\ldots,h^I)$ of adjustment functions that lead to the fulfillment of the desired properties.

3.2 Separable environments

Let the dimension of private information be $K \geq 2$. In various economic circumstances, it is natural to assume that the dimension of the allocation set coincides with the dimension of private information ($M = K$), and that for each agent $i = 1,\ldots,I$ and each $k = 1,\ldots,I$, the marginal rate of substitution between project k and the private good depends only on the k-th dimension of information. In other words, in these situations every agent is endowed with a multi-dimensional type, and each dimension of his type conveys information that is specific to one of the projects composing the social allocation.[18] We formalize the structure of this class of models in Assumption 2 presented next.

Assumption 2. *In a separable environment, $\mathcal{X} = \mathbb{R}^K_+$, for $K \geq 2$. For every agent $i = 1,\ldots,I$, every $k = 1,\ldots,K$, and every profile θ^{-i} in Θ^{-i}, the following two conditions are satisfied:*

(i) $v^i_k\big(x,\theta^{-i}\big) \;=\; \widehat{v}^i_k\big(x_k,\theta^{-i}_k\big)$, *for all x in $\mathcal{X}$.*

(ii) $C(x) \;=\; \widehat{c}_1(x_1) \;+\; \ldots \;+\; \widehat{c}_K(x_K)$, *for all x in $\mathcal{X}$.*

Condition (i) above implies that in a separable environment, each $\theta_k = (\theta^i_k,\theta^{-i}_k)$ is a type profile that captures the k-th dimension of preference heterogeneity and affects the assessment of one, and only one, component of the allocation x. When social outcome (x,t) is selected, agent i's utility is given by

$$\mathcal{U}^i(x,\theta,t^i) \;=\; \textstyle\sum_{k=1}^K \theta^i_k \widehat{v}^i_k\big(x_k,\theta^{-i}_k\big) \;-\; t^i \;\;=:\;\; \widehat{V}^i\big(x,\theta\big) \;-\; t^i.$$

[18] A similar framework has been used to study Bayesian Nash implementation in multi-dimensional screening models (Rochet and Choné (1998) and Rochet and Stole (2003)), bundling in private goods environments (Manelli and Vincent (2006)), and bundling in public goods environments (Fang and Norman (2006)).

Assumption 2 also implies, since both individual preferences and costs are separable across projects, that the efficient allocation rule satisfies a separation property: each of its component functions will be contingent solely on the corresponding dimension of information. Indeed, fix $\theta \in \Theta$ and let Assumption 2 be in place. If ψ^* is the efficient allocation rule (which, by Assumption 1, exists and is continuously differentiable), then

$$\begin{aligned}\psi^*[\theta] &= \arg\max_{x\in\mathcal{X}} \Big\{ \sum\nolimits_{i=1}^{I} \widehat{V}^i(x,\theta) \;-\; \sum\nolimits_{k=1}^{K} \widehat{c}_k(x_k) \Big\} \\ &= \arg\max_{x\in\mathcal{X}} \Big\{ \sum\nolimits_{k=1}^{K} \Big[\sum\nolimits_{i=1}^{I} \theta_k^i \widehat{v}_k^i\big(x_k, \theta_k^{-i}\big) \;-\; \widehat{c}_k(x_k) \Big] \Big\}.\end{aligned}$$

This last expression makes clear that the efficient allocation rule $\psi^* = (\psi_1^*, \ldots, \psi_K^*)$ is *separable*; i.e., for all θ in Θ, $\psi^*[\theta] \;=\; (\psi_1^*[\theta_1], \ldots, \psi_K^*[\theta_K])$, where

$$\text{(12)} \qquad \psi_k^*[\theta_k] \;=\; \arg\max_{x_k \geq 0} \Big\{ \sum\nolimits_{i=1}^{I} \theta_k^i \widehat{v}_k^i\big(x_k, \theta_k^{-i}\big) \;-\; \widehat{c}_k(x_k) \Big\}, \qquad \forall\, k = 1, \ldots, K.$$

We stress that the separability of the efficient allocation rule depends on the shape of both valuation functions and the cost of providing the allocation x. This property permits the ex post implementation of efficient decisions by means of a simple tax scheme τ. In a separable environment, it is immediate to verify that the necessary condition for the ex post implementation of ψ^* is satisfied.

Lemma 2. *Under Assumption 2, the efficient, separable allocation rule* $\psi^* \;=\; (\psi_1^*, \ldots, \psi_K^*)$*, defined in expression* (12)*, satisfies the necessary condition for ex post implementation of Corollary 1.*

Proof In a separable environment, for every $i = 1, \ldots, I$ and every $k, l = 1, \ldots, K$ with $k \neq l$, we have that

$$\frac{\partial v_k^i}{\partial x}\big(\psi^*[\theta], \theta^{-i}\big) \cdot \frac{\partial \psi^*}{\partial \theta_l^i}[\theta] \;=\; \frac{\partial \widehat{v}_k^i}{\partial x_k}\big(\psi_k^*[\theta_k], \theta_k^{-i}\big) \frac{\partial \psi_k^*}{\partial \theta_l^i}[\theta_k] \;=\; 0.$$

From the above equation one sees that expression (7) is satisfied. ■

The merit of this procedure is that it allows us to check the possibility of ex post implementation before computing the efficient allocation rule. Once this is done, the construction of the incentive tax is accomplished using the differential approach. Let us note that i's reduced form allocation under ψ^*, now denoted by $\widehat{\mathbf{v}}^i\big(\cdot;\theta^{-i},\psi^*\big)$, is in this case also separable. That is, for all $i = 1, \ldots, I$ and all θ^{-i} in Θ^{-i},

$$\text{(13)} \qquad \widehat{\mathbf{v}}^i\big(\theta^i;\theta^{-i},\psi^*\big) \;=\; \big(\, \widehat{\mathbf{v}}_k^i\big(\theta_k^i;\theta_k^{-i},\psi_k^*\big)\,\big)_{k=1}^{K} \;:=\; \big(\, \widehat{v}_k^i\big(\psi_k^*[\theta_k^i,\theta_k^{-i}],\theta_k^{-i}\big)\,\big)_{k=1}^{K}.$$

From the above relation it is immediate to verify that condition (ii) of Proposition 1 implies that $\widehat{\mathbf{v}}_k^i\big(\cdot;\theta_k^{-i},\psi_k^*\big)$ is increasing in θ_k^i, for every $i = 1, \ldots, I$, $k = 1, \ldots, K$, and every

θ_k^{-i}. To see this, take any two elements θ^i and $\hat{\theta}^i$ of Θ^i with all components equal to zero, except for θ_k^i and $\hat{\theta}_k^i$, respectively, and apply condition (ii). Thus, in effect, the hypotheses of a separable environment transform a K-dimensional implementation problem into K different one-dimensional problems, for which the solution is known from Section 3.1. We work with a separable tax scheme $\tau = (\tau^1, \ldots, \tau^I)$, where i's tax burden is given by

$$\tau^i[\theta] \;=\; \sum\nolimits_{k=1}^{K} \tau_k^i[\theta_k], \qquad\qquad \forall\, \theta \in \Theta.$$

Let $\Gamma = (\psi^*, \tau)$ be a direct mechanism such that ψ^* is the separable, efficient allocation rule satisfying the monotonicity condition of Proposition 1 and τ is a separable tax system. Fix a type profile θ^{-i} for all agents other than i. Under Assumption 2, i's ex post incentive constraint takes the following form: for any two θ^i, $\hat{\theta}^i$ in Θ^i,

$$\begin{aligned} U^i\big(\theta^i;\theta^{-i},\Gamma^*\big) \;&\geq\; \overline{U}^i\big(\hat{\theta}^i,\theta^i;\theta^{-i},\Gamma^*\big) \\ &=\; \widehat{V}^i\big(\psi^*[\hat{\theta}^i,\theta^{-i}],\theta\big) \;-\; \tau^i[\hat{\theta}^i,\theta^{-i}] \\ &=\; \sum\nolimits_{k=1}^{K} \Big\{\theta_k^i \widehat{v}_k^i\big(\psi_k^*[\hat{\theta}_k^i,\theta_k^{-i}],\theta_k^{-i}\big) \;-\; \tau_k^i[\hat{\theta}_k^i,\theta_k^{-i}]\Big\}. \end{aligned}$$

Clearly, i's ex post incentive constraints will be satisfied provided

$$\theta_k^i \widehat{v}_k^i\big(\psi_k^*[\theta_k^i,\theta_k^{-i}],\theta_k^{-i}\big) \;-\; \tau_k^i[\theta_k^i,\theta_k^{-i}] \;\geq\; \theta_k^i \widehat{v}_k^i\big(\psi_k^*[\hat{\theta}_k^i,\theta_k^{-i}],\theta_k^{-i}\big) \;-\; \tau_k^i[\hat{\theta}_k^i,\theta_k^{-i}], \qquad \forall\, k.$$

We apply the differential approach, as in the one-dimensional case, to obtain the separable tax scheme τ that implements $\psi^* = (\psi_1^*, \ldots, \psi_K^*)$ ex post. For each type profile θ in Θ, agent i's tax schedule $\tau^i[\theta] = \sum_k \tau_k^i[\theta_k]$ is given by

$$\tau_k^i[\theta_k] \;=\; \theta_k^i \widehat{v}_k^i\big(\psi_k^*[\theta_k],\theta_k^{-i}\big) \;-\; \int_0^{\theta_k^i} \widehat{\mathbf{v}}_k^i\big(\tilde{\theta}_k^i;\theta_k^{-i},\psi_k^*\big)\,\mathrm{d}\tilde{\theta}_k^i \;+\; h_k^i(\theta_k^{-i}). \tag{14}$$

These surplus extraction incentive taxes are derived using local necessary conditions from the differential approach. As in the one-dimensional case, it remains to check that the direct mechanism $\Gamma^* = (\psi^*, \tau)$ is indeed efficient, ex post incentive compatible, i.e., satisfies all the ex post incentive compatibility constraints.

Proposition 3. *In a separable environment where Assumption 2 is satisfied, let* $\psi^* = (\psi_1^*, \ldots, \psi_K^*)$ *be an efficient, separable, continuously differentiable allocation rule as defined in* (12) *such that, for all* $i = 1, \ldots, I$*, and all* θ^{-i}*, the reduced form allocation* $\widehat{\mathbf{v}}^i\big(\cdot;\theta^{-i},\psi^*\big)$ *given in expression* (13) *is monotone. Then* ψ^* *is ex post implementable by the separable tax scheme* τ *if and only if for all* $i = 1, \ldots, I$*, all* $k = 1, \ldots, K$ *and all* θ_k*,* $\tau_k^i[\theta_k]$ *is given by expression* (14)*.*

Proof It has been already shown that if ψ^* is ex post implementable by a separable transfer scheme $\tau = (\tau^1, \ldots, \tau^I)$, then (14) must be satisfied. For sufficiency, we have argued that if ψ^* is an efficient, separable allocation rule and τ a separable transfer scheme, then i's ex post incentive constraints will be satisfied provided one shows that for all $k = 1, \ldots, K$, for every type profile θ_k^{-i}, and for any two θ_k^i and $\hat{\theta}_k^i$,

$$\Big\{\theta_k^i \widehat{v}_k^i\big(\psi_k^*[\theta_k], \theta_k^{-i}\big) \;-\; \tau_k^i[\theta_k]\Big\} \;-\; \Big\{\theta_k^i \widehat{v}_k^i\big(\psi_k^*[\hat{\theta}_k^i, \theta_k^{-i}], \theta_k^{-i}\big) \;-\; \tau_k^i[\hat{\theta}_k^i, \theta_k^{-i}]\Big\} \;\geq\; 0.$$

Since the reduced form allocation $\widehat{\mathbf{v}}^i\big(\cdot; \theta^{-i}, \psi^*\big)$ is monotone, this last step can be verified using the same arguments of Proposition 2. Thus, the separable transfer scheme τ implements ψ^* ex post, as desired. ■

We modify Example 1 to illustrate the differential approach in a separable environment.

Example 2. The set of agents in the economy is $I = \{1, 2, 3\}$; for each $j \in I$, let $\Theta^j = [0, \overline{\theta}_1^j] \times [0, \overline{\theta}_2^j]$. The set of social alternatives is now $\mathcal{X} = \mathbb{R}_+^2$, so $K = M = 2$. As before, only agent 1's type is payoff-relevant to other agents. Preferences for social alternatives are separable:

$$\begin{aligned} \widehat{V}^1\big(x, \theta^1\big) &= \theta_1^1\big(\alpha_1^1 x_1\big) \;+\; \theta_2^1\big(\beta_2^1 x_2\big); \\ \widehat{V}^i\big(x, \theta^1, \theta^i\big) &= \theta_1^i\big(\theta_1^1 \alpha_1^i x_1\big) \;+\; \theta_2^i\big(\theta_2^1 \beta_2^i x_2\big), \qquad\qquad i = 2, 3; \end{aligned}$$

where, α_k^j and β_k^j are known, positive parameters. We maintain the quadratic, separable cost function: $C(x) = \frac{1}{2}(x_1^2 + x_2^2)$. The central planner uses a direct mechanism $\Gamma^* = (\psi^*, \tau)$ to maximize social welfare $\sum_j \widehat{V}^j\big(x, \theta\big) - C(x)$.

In this example, the efficient, separable allocation rule $\psi^* = (\psi_1^*, \psi_2^*)$ is defined on Θ by:

(15) $$\psi_1^*[\theta_1] = \theta_1^1\big(\alpha_1^1 + \alpha_1^2\theta_1^2 + \alpha_1^3\theta_1^3\big),$$

(16) $$\psi_2^*[\theta_2] = \theta_2^1\big(\beta_2^1 + \beta_2^2\theta_2^2 + \beta_2^3\theta_2^3\big).$$

Observe that the monotonicity of the reduced form allocations generated by ψ^* is satisfied. We make use of expression (14) to construct the separable, surplus extraction tax scheme to implement ψ^* ex post. Suppose that agents 2 and 3 are following their equilibrium strategies and truthfully reporting their types. Ignoring the constants of integration, agent 1's tax schedule, $\tau^1[\theta] \;=\; \tau_1^1[\theta_1] \;+\; \tau_2^1[\theta_2]$, is define as follows:

$$\begin{aligned} \tau_1^1[\theta_1] &= \theta_1^1\alpha_1^1\big\{\theta_1^1\big(\alpha_1^1 + \alpha_1^2\theta_1^2 + \alpha_1^3\theta_1^3\big)\big\} \;-\; \int_0^{\theta_1^1} \alpha_1^1\big\{\tilde{\theta}_1^1\big(\alpha_1^1 + \alpha_1^2\theta_1^2 + \alpha_1^3\theta_1^3\big)\big\}\, \mathrm{d}\tilde{\theta}_1^1; \\ \tau_2^1[\theta_2] &= \theta_2^1\beta_2^1\big\{\theta_2^1\big(\beta_2^1 + \beta_2^2\theta_2^2 + \beta_2^3\theta_2^3\big)\big\} \;-\; \int_0^{\theta_2^1} \beta_2^1\big\{\tilde{\theta}_2^1\big(\beta_2^1 + \beta_2^2\theta_2^2 + \beta_2^3\theta_2^3\big)\big\}\, \mathrm{d}\tilde{\theta}_2^1. \end{aligned}$$

After some manipulations, we find

(17) $$\tau_1^1[\theta_1] = \frac{1}{2}\big(\theta_1^1\big)^2\alpha_1^1\big\{\alpha_1^1 + \alpha_1^2\theta_1^2 + \alpha_1^3\theta_1^3\big\};$$

(18) $$\tau_2^1[\theta_2] = \frac{1}{2}\big(\theta_2^1\big)^2\beta_2^1\big\{\beta_2^1 + \beta_2^2\theta_2^2 + \beta_2^3\theta_2^3\big\}.$$

Given the efficient allocation rule $\psi^* = (\psi_1^*, \psi_2^*)$ defined in (15)–(16) and taxes $\tau^1 = \tau_1^1 + \tau_2^1$ defined in (17)–(18), agent 1's payoff when his type is θ^1 and his report is $\hat{\theta}^1$, and agents 2 and 3 are truthfully reporting, equals:

$$\begin{aligned}\overline{U}^1\big(\hat{\theta}^1, \theta^1; \theta^{-1}, \Gamma^*\big) \;&=\; \theta_1^1\alpha_1^1\hat{\theta}_1^1\big(\alpha_1^1 + \alpha_1^2\theta_1^2 + \alpha_1^3\theta_1^3\big) \;+\; \theta_2^1\beta_2^1\hat{\theta}_2^1\big(\beta_2^1 + \beta_2^2\theta_2^2 + \beta_2^3\theta_2^3\big) \\ &\quad - \frac{1}{2}\Big[\big(\hat{\theta}_1^1\big)^2\alpha_1^1\big\{\alpha_1^1 + \alpha_1^2\theta_1^2 + \alpha_1^3\theta_1^3\big\} \;+\; \big(\hat{\theta}_2^1\big)^2\beta_2^1\big\{\beta_2^1 + \beta_2^2\theta_2^2 + \beta_2^3\theta_2^3\big\}\Big].\end{aligned}$$

One can readily see from the above expression that the optimal report for agent 1 is $(\hat{\theta}_1^1, \hat{\theta}_2^1) = (\theta_1^1, \theta_2^1)$. $\square$

We shall mention that in the above example, the set of parameters for which efficient, ex post implementation is possible, is open (since no restrictions were imposed on the values of parameters α_k^j and β_k^j other than being strictly greater than zero). More generally, in separable environments there is no need to impose further restrictions to achieve efficient, ex post implementation with interdependent valuations. This contrast with the findings of Jehiel and Moldovanu (2001), who present an impossibility result for efficient, Bayesian Nash implementation (hence ex post implementation) in framework with valuations that are linear in all types, and a finite number of alternatives, $k = 1 \ldots, K$. Agent j receives a signal θ_k^j that affects valuations for alternative k; i.e., if alternative k is selected, then agent i obtains

$$V_k^i(\theta_k^1, \ldots, \theta_k^I) \;=\; \sum\nolimits_{j=1}^{I} \alpha_{ik}^j \theta_k^j, \qquad\qquad k = 1, \ldots, K;$$

where α_{ik}^j are known parameters. Within this framework, Jehiel and Moldovanu (2001) show that efficient, Bayesian Nash implementation is possible only if a non-generic condition on parameters is satisfied.[19]

The main difference between their result and ours is that, despite the separability in valuation functions, the efficient allocation rule in Jehiel and Moldovanu (2001) is not separable: the probability of k being chosen as the efficient alternative also depends on signals $\theta_l^1, \ldots, \theta_l^I$, for $l \neq k$. This is key to our positive result, which relies on the fact that Assumption 2 transforms a K-dimensional incentive problem into K separated, one-dimensional problems. Any modification in the framework that breaks down this separation property is bound to bring further restrictions back. We modify the previous example to illustrate this point.

Example 3. The setup is as in Example 2, except for the presence of a exogenous resource constraint of the form $x_1 + x_2 \leq L$ (for some positive L). To find the constrained efficient

[19] Namely, for all agents i and all alternatives k, l,

$$\frac{\alpha_{ik}^i}{\alpha_{il}^i} \;=\; \frac{\sum_{j=1}^{I} \alpha_{jk}^i}{\sum_{j=1}^{I} \alpha_{jl}^i}.$$

The above ratios are called i's private and social rates of information substitution, respectively. This congruence condition is non-generic in that the set of parameters for which it is satisfied is closed and has Lebesgue measure zero.

allocation, the central planner maximizes social welfare subject to $x_1 + x_2 \leq L$. We focus our attention on the solution to this program that has both x_1 and x_2 positive and its sum equal to L. In such case, the efficient allocation rule $\psi^* = (\psi_1^*, \psi_2^*)$ is defined by:

$$\psi_1^*[\theta] = \frac{L}{2} + \frac{1}{2}\{\theta_1^1(\alpha_1^1 + \alpha_1^2\theta_1^2 + \alpha_1^3\theta_1^3) - \theta_2^1(\beta_2^1 + \beta_2^2\theta_2^2 + \beta_2^3\theta_2^3)\}; \tag{19}$$

$$\psi_2^*[\theta] = \frac{L}{2} + \frac{1}{2}\{\theta_2^1(\beta_2^1 + \beta_2^2\theta_2^2 + \beta_2^3\theta_2^3) - \theta_1^1(\alpha_1^1 + \alpha_1^2\theta_1^2 + \alpha_1^3\theta_1^3)\}. \tag{20}$$

The constrained efficient allocation rule ψ^* is obviously not separable, so we cannot apply Proposition 3. Additional restrictions (congruent with the necessary condition of Corollary 1) must be satisfied. Given ψ^* described in equations (19)–(20), agent 1's payoff when his type is θ^1 and his report is $\hat{\theta}^1$, and agents 2 and 3 are truthfully reporting, is given by

$$\widehat{V}^1(\psi^*[\hat{\theta}^1, \theta^{-1}], \theta^1) - \tau^1[\hat{\theta}^1, \theta^{-1}] = \theta_1^1\alpha_1^1\psi_1^*[\hat{\theta}^1, \theta^{-1}] + \theta_2^1\beta_2^1\psi_2^*[\hat{\theta}^1, \theta^{-1}] - \tau^1[\hat{\theta}^1, \theta^{-1}].$$

Applying first order conditions to the above expression, we obtain the following system of differential equations describing agent 1's incentive constraints:

$$\frac{\partial \tau^1}{\partial \hat{\theta}_1^1}[\hat{\theta}^1, \theta^{-1}] = \frac{\alpha_1^1 + \alpha_1^2\theta_1^2 + \alpha_1^3\theta_1^3}{2}\{\theta_1^1\alpha_1^1 - \theta_2^1\beta_2^1\};$$

$$\frac{\partial \tau^1}{\partial \hat{\theta}_2^1}[\hat{\theta}^1, \theta^{-1}] = \frac{\beta_2^1 + \beta_2^2\theta_2^2 + \beta_2^3\theta_2^3}{2}\{\theta_2^1\beta_2^1 - \theta_1^1\alpha_1^1\}.$$

The appropriate incentive taxes τ^1 are found by integrating the above system with $\hat{\theta}^1 = \theta^1$ as an optimal strategy for agent 1. This is possible only if

$$\frac{\alpha_1^1}{\beta_2^1} = \frac{\alpha_1^1 + \alpha_1^2\theta_1^2 + \alpha_1^3\theta_1^3}{\beta_2^1 + \beta_2^2\theta_2^2 + \beta_2^3\theta_2^3}. \quad \square$$

3.3 Quasi-separable environments

The crucial component of the positive result in separable environments is the possibility of transforming a multi-dimensional incentive problem into several, one-dimensional problems. Assumption 2 states sufficient conditions to obtain a separable, efficient allocation rule. We now consider a broader class of environments for which ex post implementation of efficient allocation rules is still achievable by means of simple tax schemes. These environments are not completely separable: some components of the efficient allocation rule will be contingent on more than one dimension of information.

Recall that agent i's type space is $\Theta^i = \times_{k=1}^K[0, \overline{\theta}_k^i]$. For simplicity, we assume that the dimension of the allocation set $\mathcal{X}$ is $M = K + 1$, and denote any social alternative x in $\mathcal{X} = \mathbb{R}_+^{K+1}$ by $x = (x_1, \ldots, x_K, x_{K+1})$. We deal with valuations for social alternatives that are quasi-separable: for $k = 1, \ldots, K$, the assessment of project k depends on the k-th dimension of private information, and the assessment of project $K+1$ depends on all dimensions of information. Furthermore, the interaction between the different dimensions of preference heterogeneity and project $K+1$ takes a specific form: each dimension has the same effect on the appraisal of x_{K+1}. Several economic circumstances can be studied employing this class of environments, which we call *quasi-separable*. For example, in the public good application mentioned in the introduction, θ_1^i captures the appreciation for science and interacts with x_1, the science facility in the educational institution; θ_2^i captures the appreciation for arts and interacts with x_2, the arts facility; and both dimensions of private information affect the appraisal of x_3, a general purpose facility. The following key hypothesis is maintained throughout this subsection.

Assumption 3. *In a quasi-separable environment, the dimension of the allocation set is $M = K+1$, so that $\mathcal{X} = \mathbb{R}_+^{K+1}$. For all i, let ζ^i be the function defined on Θ^i by*

$$\zeta^i(\theta^i) \;=\; \theta_1^i \;+\; \ldots \;+\; \theta_K^i,$$

and let $\zeta(\theta)$ denote the profile $(\zeta^1(\theta^1), \ldots, \zeta^I(\theta^I))$. For every agent $i = 1, \ldots, I$, every dimension of private information $k = 1, \ldots, K$, and every type profile θ^{-i} in Θ^{-i}, the following two conditions are satisfied:

(i) $v_k^i\big(x, \theta^{-i}\big) \;=\; \widetilde{v}_k^i\big(x_k, \theta_k^{-i}\big) \;+\; \widetilde{w}^i\big(x_{K+1}, \zeta^{-i}(\theta^{-i})\big)$*, for all x in $\mathcal{X}$.*

(ii) $C(x) \;=\; \widetilde{c}_1(x_1) \;+\; \ldots \;+\; \widetilde{c}_{K+1}(x_{K+1})$*, for all x in $\mathcal{X}$.*

There are several things to observe from Assumption 3. First, each appraisal function v_k^i is contingent only on projects x_k and x_{K+1} in an additive way; second, and the assessment of x_{K+1} is uniform across all dimensions of information; third, due to the presence of the aggregator functions ζ^j, marginal variations in any dimension of private information held by any agent have the same impact on i's assessment of project x_{K+1}. In a quasi-separable environment, agent i's utility is given by:

$$\begin{aligned}
\mathcal{U}^i(x, \theta, t^i) \;&=\; \sum\nolimits_{k=1}^{K} \theta_k^i \Big\{ \widetilde{v}_k^i\big(x_k, \theta_k^{-i}\big) \;+\; \widetilde{w}^i\big(x_{K+1}, \zeta^{-i}(\theta^{-i})\big) \Big\} \;-\; t^i \\
&=\; \sum\nolimits_{k=1}^{K} \theta_k^i \widetilde{v}_k^i\big(x_k, \theta_k^{-i}\big) \;+\; \zeta^i(\theta^i)\widetilde{w}^i\big(x_{K+1}, \zeta^{-i}(\theta^{-i})\big) \;-\; t^i \\
&=:\; \widetilde{V}^i\big(x, \theta\big) \;-\; t^i.
\end{aligned}$$

An implication of Assumption 3 is that the efficient allocation rule will be *quasi-separable* in the following sense: the first K component functions will be contingent solely on the corresponding dimension of information; the last $K+1$ component function will depend on all dimensions of information through the aggregator functions. Fix θ and let ψ^* be the efficient allocation rule. Then

$$\begin{aligned}
\psi^*[\theta] \;&=\; \arg\max_{x \in \mathcal{X}} \Big\{ \sum\nolimits_{i=1}^{I} \widetilde{V}^i\big(x, \theta\big) \;-\; \sum\nolimits_{k=1}^{K+1} \widetilde{c}_k(x_k) \Big\} \\
&=\; \arg\max_{x \in \mathcal{X}} \Big\{ \sum\nolimits_{k=1}^{K} \Big[\sum\nolimits_{i=1}^{I} \theta_k^i \widetilde{v}_k^i\big(x_k, \theta_k^{-i}\big) \;-\; \widetilde{c}_k(x_k) \Big] \\
&\qquad\qquad +\; \sum\nolimits_{i=1}^{I} \zeta^i(\theta^i)\widetilde{w}^i\big(x_{K+1}, \zeta^{-i}(\theta^{-i})\big) \;-\; \widetilde{c}_{K+1}(x_{K+1}) \Big\}.
\end{aligned}$$

Thus, under Assumption 3, the efficient allocation rule ψ^* is quasi-separable, with its last component function ψ^*_{K+1} depending on θ^i through the aggregator function ζ^i:

$$(21) \qquad \psi^*[\theta] \;=\; \big(\psi^*_1[\theta_1], \ldots, \psi^*_K[\theta_K], \psi^*_{K+1}[\zeta(\theta)]\big)\;, \qquad \forall\; \theta \in \Theta;$$

where

$$\psi^*_k[\theta_k] \;=\; \arg\max_{x_k \geq 0} \Big\{ \sum\nolimits_{i=1}^{I} \theta^i_k \widetilde{v}^i_k\big(x_k, \theta^{-i}_k\big) \;-\; \widetilde{c}_k(x_k) \Big\}, \qquad k = 1, \ldots, K;$$
$$\psi^*_{K+1}[\zeta(\theta)] \;=\; \arg\max_{x_{K+1} \geq 0} \Big\{ \sum\nolimits_{i=1}^{I} \zeta^i(\theta^i) \widetilde{w}^i\big(x_{K+1}, \zeta^{-i}(\theta^{-i})\big) \;-\; \widetilde{c}_{K+1}(x_{K+1}) \Big\}.$$

As in previous case, before proceeding with the construction of the appropriate tax system, we verify that the necessary condition for ex post implementation of ψ^* is satisfied. From Corollary 1, we must check that for all type profiles θ^{-i}, for all dimensions of information $k, l = 1, \ldots, K$, the following condition is fulfilled almost everywhere on Θ^i:

$$\frac{\partial v^i_k}{\partial x}\big(\psi^*[\theta], \theta^{-i}\big) \cdot \frac{\partial \psi^*}{\partial \theta^i_l}[\theta] \;=\; \frac{\partial v^i_l}{\partial x}\big(\psi^*[\theta], \theta^{-i}\big) \cdot \frac{\partial \psi^*}{\partial \theta^i_k}[\theta].$$

This condition is obviously satisfied if $k = l$. For $k \neq l$, observe that in a quasi-separable environment, renaming coordinates as necessary, we have

$$\frac{\partial v^i_k}{\partial x} \cdot \frac{\partial \psi^*}{\partial \theta^i_l} \;=\; \left(0, \ldots, \frac{\partial \widetilde{v}^i_k}{\partial x_k}, 0, \ldots, \frac{\partial \widetilde{w}^i}{\partial x_{K+1}}\right) \cdot \left(0, \ldots, 0, \frac{\partial \psi^*_l}{\partial \theta^i_l}, \ldots, \frac{\partial \psi^*_{K+1}}{\partial \zeta^i} \frac{\partial \zeta^i}{\partial \theta^i_l}\right)$$

and

$$\frac{\partial v^i_l}{\partial x} \cdot \frac{\partial \psi^*}{\partial \theta^i_k} \;=\; \left(0, \ldots, 0, \frac{\partial \widetilde{v}^i_l}{\partial x_l}, \ldots, \frac{\partial \widetilde{w}^i}{\partial x_{K+1}}\right) \cdot \left(0, \ldots, \frac{\partial \psi^*_k}{\partial \theta^i_k}, 0, \ldots, \frac{\partial \psi^*_{K+1}}{\partial \zeta^i} \frac{\partial \zeta^i}{\partial \theta^i_k}\right).$$

Thus, the necessary condition for ex post implementation of the efficient allocation rule ψ^* will be satisfied provided

$$\frac{\partial \zeta^i(\theta^i)}{\partial \theta^i_k} \;=\; \frac{\partial \zeta^i(\theta^i)}{\partial \theta^i_l}\;, \qquad \forall\; k, l = 1, \ldots, K.$$

This is guaranteed by the definition of the aggregator function ζ^i. In other words, any two dimensions of information have the same marginal impact on the assessment of project x_{K+1}. Thus, in quasi-separable environments the necessary condition for ex post implementation of efficient allocation rules is satisfied.

Lemma 3. *If Assumption 3 is satisfied, the efficient allocation rule $\psi^* = (\psi^*_1, \ldots, \psi^*_K, \psi^*_{K+1})$ defined in expression* (21) *satisfies the necessary condition for ex post implementation of Corollary 1.*

In a quasi-separable environment, agent i's reduced form allocation generated by ψ^*, now denoted $\widetilde{\mathbf{v}}^i(\cdot;\theta^{-i},\psi^*)$, which by Proposition 1 must be monotone, takes the following form: for all θ^{-i} in Θ^{-i}, and all θ^i in Θ^i,

$$\begin{aligned}(22)\qquad \widetilde{\mathbf{v}}^i(\theta^i;\theta^{-i},\psi^*) &= \left(\widetilde{\mathbf{v}}^i_k(\theta^i_k;\theta^{-i}_k,\psi^*_k) + \widetilde{\mathbf{w}}^i(\zeta^i(\theta^i);\zeta^{-i}(\theta^{-i}),\psi^*_{K+1}) \right)_{k=1}^K \\ &:= \left(\widetilde{v}^i_k(\psi^*_k[\theta_k],\theta^{-i}_k) + \widetilde{w}^i(\psi^*_{K+1}[\zeta(\theta)],\zeta^{-i}(\theta^{-i})) \right)_{k=1}^K .\end{aligned}$$

Note that condition (ii) of Proposition 1 does not necessarily imply the monotonicity of each component function of the efficient allocation rule ψ^*, although if ψ^*_k is monotone, for all $k=1,\ldots,K+1$, then the reduced form allocation $\widetilde{\mathbf{v}}^i(\cdot;\theta^{-i},\psi^*)$ is also monotone. We now make use of the differential approach to construct an incentive tax scheme that implements the efficient allocation rule ψ^* as an ex post equilibrium. To that purpose, let $\Gamma^*=(\psi^*,\tau)$ be an efficient, direct mechanism; fix a type profile θ^{-i} in Θ^{-i}, and suppose all agents other than i are truthfully announcing their types. For any θ^i and $\hat{\theta}^i$ that belong to Θ^i, agent i's utility when his actual type is θ^i and his report to the central planner is $\hat{\theta}^i$, is given by:

$$\begin{aligned}\overline{U}^i(\hat{\theta}^i,\theta^i;\theta^{-i},\Gamma^*) &= \widetilde{V}^i(\psi^*[\hat{\theta}^i,\theta^{-i}],\theta) - \tau^i[\hat{\theta}^i,\theta^{-i}] \\ &= \sum_{k=1}^K \theta^i_k\widetilde{v}^i_k(\psi^*_k[\hat{\theta}^i_k,\theta^{-i}_k],\theta^{-i}_k) \\ &\quad + \zeta^i(\theta^i)\widetilde{w}^i(\psi^*_{K+1}[\zeta^i(\hat{\theta}^i),\zeta^{-i}(\theta^{-i})],\zeta^{-i}(\theta^{-i})) - \tau^i[\hat{\theta}^i,\theta^{-i}].\end{aligned}$$

Using the differential approach, we find that the first order conditions for ex post incentive compatibility are given by the relation

$$\begin{aligned}0 &= \theta^i_k\frac{\partial\widetilde{v}^i_k}{\partial x_k}(\psi^*_k[\theta_k],\theta^{-i}_k)\frac{\partial\psi^*_k}{\partial\theta^i_k}[\theta_k] \\ &\quad + \zeta^i(\theta^i)\frac{\partial\widetilde{w}^i}{\partial x_{K+1}}(\psi^*_{K+1}[\zeta(\theta)],\zeta^{-i}(\theta^i))\frac{\partial\psi^*_{K+1}}{\partial\zeta^i}[\zeta(\theta)]\frac{\partial\zeta^i}{\partial\theta^i_k}(\theta^i) - \frac{\partial\tau^i}{\partial\theta^i_k}[\theta];\end{aligned}$$

which should be satisfied for all dimensions of private information $k=1,\ldots,K$. Thus, taxes for agent i are defined as a solution to the following system of differential equations:

$$\begin{aligned}\frac{\partial\tau^i}{\partial\theta^i_k}[\theta] &= \theta^i_k\frac{\partial\widetilde{v}^i_k}{\partial x_k}\frac{\partial\psi^*_k}{\partial\theta^i_k}[\theta_k] + \zeta^i(\theta^i)\frac{\partial\widetilde{w}^i}{\partial x_{K+1}}\frac{\partial\psi^*_{K+1}}{\partial\zeta^i}\frac{\partial\zeta^i}{\partial\theta^i_k}(\theta^i) \\ &= \theta^i_k\frac{\partial\widetilde{v}^i_k}{\partial x_k}\frac{\partial\psi^*_k}{\partial\theta^i_k}[\theta_k] + \widetilde{v}^i_k(\psi^*_k[\theta_k],\theta^{-i}_k) - \widetilde{v}^i_k(\psi^*_k[\theta_k],\theta^{-i}_k) \\ &\quad + \zeta^i(\theta^i)\frac{\partial\widetilde{w}^i}{\partial x_{K+1}}\frac{\partial\psi^*_{K+1}}{\partial\zeta^i}\frac{\partial\zeta^i}{\partial\theta^i_k}(\theta^i) + \widetilde{w}^i(\psi^*_{K+1}[\zeta(\theta)],\zeta^{-i}(\theta^{-i}))\frac{\partial\zeta^i}{\partial\theta^i_k}(\theta^i) \\ &\quad - \widetilde{w}^i(\psi^*_{K+1}[\zeta(\theta)],\zeta^{-i}(\theta^{-i}))\frac{\partial\zeta^i}{\partial\theta^i_k}(\theta^i), \qquad \forall\ k=1,\ldots,K.\end{aligned}$$

The function ζ^i has equal partial derivatives with respect to every component. This feature makes the recovery of the tax schedule τ^i from the system of partial differential equations possible. The above system yields a solution of the form:

$$(23)\qquad \tau^i[\theta] \;=\; \sum\nolimits_{k=1}^{K} \Big\{ \theta_k^i \widetilde{v}_k^i\big(\psi_k^*[\theta_k], \theta_k^{-i}\big) \;-\; \int_0^{\theta_k^i} \widetilde{\mathbf{v}}_k^i\big(\tilde{\theta}_k^i; \theta_k^{-i}, \psi_k^*\big)\, \mathrm{d}\tilde{\theta}_k^i \;+\; h_k^i(\theta_k^{-i}) \Big\}$$
$$+\; \zeta^i(\theta^i)\widetilde{w}^i\big(\psi_{K+1}^*[\zeta(\theta)], \zeta^{-i}(\theta^{-i})\big)$$
$$-\; \int_0^{\zeta^i(\theta^i)} \widetilde{\mathbf{w}}^i\big(\tilde{\zeta}^i; \zeta^{-i}(\theta^{-i}), \psi_{K+1}^*\big)\, \mathrm{d}\tilde{\zeta}^i \;+\; h_{K+1}^i(\zeta^{-i}(\theta^{-i})).$$

Agent i's tax burden is constructed to guarantee truth-telling as an ex post equilibrium of the game induced by Γ^*. Under Assumption 3, the first K components of i's valuation function $\widetilde{V}^i$ are separable: each k-th component is contingent on project x_k and types θ_k alone ($k = 1, \ldots, K$). Moreover, since the cost function is separable, the k-th component of the efficient allocation rule, ψ_k^*, depends solely on $\theta_k = (\theta_k^1, \ldots, \theta_k^I)$. On the other hand, the assessment of x_{K+1} is contingent on all dimensions of private information. However, this dependency takes a particular form: every dimension of information has the same marginal effect on the appraisal of x_{K+1}, and thus we can use the aggregator functions $\zeta = (\zeta^1, \ldots, \zeta^I)$ to compute the efficient allocation ψ_{K+1}^* as a function of ζ. The combination of both separability and aggregation makes ex post implementation possible. Taxes paid by agent i reflect this structure. Each of the first K terms of τ^i is a surplus extraction tax aimed at solving the incentive problem related to the efficient allocation of x_k. The last term has a similar form but, in turn, depends on all dimensions of information via the aggregator functions. The logic exposed before prevails: taxes paid by agent i are designed to extract all surplus generated by the efficient allocation, safe for the informational rent needed to ensure truth-telling. We can now establish the following result.

Proposition 4. *In a quasi-separable environment where Assumption 3 is satisfied, let* $\psi^* = (\psi_1^*, \ldots, \psi_K^*, \psi_{K+1}^*)$ *be an efficient, quasi-separable allocation rule as defined in* (21) *such for all* $i = 1, \ldots, I$*, and all* θ^{-i} *in* Θ^{-i}*, the reduced form allocation* $\widetilde{\mathbf{v}}^i\big(\cdot; \theta^{-i}, \psi^*\big)$ *given in expression* (22) *is monotone. Then* ψ^* *is ex post implementable by the quasi-separable tax scheme* τ *if and only if for all* $i = 1, \ldots, I$*, and all* θ*,* $\tau^i(\theta)$ *is given by expression* (23)*.*

Proof We have already established, using local necessary conditions, that if ψ^* is ex post implementable, then the appropriate quasi-separable tax scheme is of the form given in (23). For sufficiency, we show that the surplus extraction tax scheme τ of expression (23) implements ψ^* ex post. To that extend, write $\Gamma^* = (\psi^*, \tau)$, fix θ^{-i} in Θ^{-i} and compute, for any two θ^i and $\hat{\theta}^i$, $\overline{U}^i\big(\hat{\theta}^i, \theta^i; \theta^{-i}, \Gamma^*\big)$ and $U^i\big(\theta^i; \theta^{-i}, \Gamma^*\big)$:

$$\overline{U}^i\big(\hat{\theta}^i, \theta^i; \theta^{-i}, \Gamma^*\big) \;=\; \sum\nolimits_{k=1}^{K} \Big\{ \big(\theta_k^i - \hat{\theta}_k^i\big)\widetilde{v}_k^i\big(\psi_k^*[\hat{\theta}_k^i, \theta_k^{-i}], \theta_k^{-i}\big) \;+\; \int_0^{\hat{\theta}_k^i} \widetilde{\mathbf{v}}_k^i\big(\tilde{\theta}_k^i; \theta_k^{-i}, \psi_k^*\big)\, \mathrm{d}\tilde{\theta}_k^i \Big\}$$
$$+\; \big\{\zeta^i(\theta^i) - \zeta^i(\hat{\theta}^i)\big\}\widetilde{w}^i\big(\psi_{K+1}^*[\zeta^i(\hat{\theta}^i), \zeta^{-i}(\theta^{-i})], \zeta^{-i}(\theta^{-i})\big)$$
$$+\; \int_0^{\zeta^i(\hat{\theta}^i)} \widetilde{\mathbf{w}}^i\big(\tilde{\zeta}^i; \zeta^{-i}(\theta^{-i}), \psi_{K+1}^*\big)\, \mathrm{d}\tilde{\zeta}^i;$$

and

$$U^i\big(\theta^i;\theta^{-i},\Gamma^*\big) \;=\; \sum_{k=1}^{K}\Big\{\int_0^{\theta^i_k}\widetilde{\mathbf{v}}^i_k\big(\tilde{\theta}^i_k;\theta^{-i}_k,\psi^*_k\big)\,\mathrm{d}\tilde{\theta}^i_k\Big\} \;+\; \int_0^{\zeta^i(\theta^i)}\widetilde{\mathbf{w}}^i\big(\tilde{\zeta}^i;\zeta^{-i}(\theta^{-i}),\psi^*_{K+1}\big)\,\mathrm{d}\tilde{\zeta}^i.$$

Now observe that

$$\begin{aligned}
U^i\big(\theta^i;\theta^{-i},\Gamma^*\big) \;&-\; \overline{U}^i\big(\hat{\theta}^i,\theta^i;\theta^{-i},\Gamma^*\big)\\
&=\; \sum_{k=1}^{K}\Big\{\int_{\hat{\theta}^i_k}^{\theta^i_k}\Big[\widetilde{\mathbf{v}}^i_k\big(\tilde{\theta}^i_k;\theta^{-i}_k,\psi^*_k\big) \;-\; \widetilde{\mathbf{v}}^i_k\big(\hat{\theta}^i_k;\theta^{-i}_k,\psi^*_k\big)\Big]\,\mathrm{d}\tilde{\theta}^i_k\Big\}\\
&\quad+\; \int_{\zeta^i(\hat{\theta}^i)}^{\zeta^i(\theta^i)}\Big[\widetilde{\mathbf{w}}^i\big(\tilde{\zeta}^i;\zeta^{-i}(\theta^{-i}),\psi^*_{K+1}\big) \;-\; \widetilde{\mathbf{w}}^i\big(\zeta^i(\hat{\theta}^i);\zeta^{-i}(\theta^{-i}),\psi^*_{K+1}\big)\Big]\,\mathrm{d}\tilde{\zeta}^i\\
&\geq\; 0,
\end{aligned}$$

where the last inequality follows from the monotonicity of i's reduced form allocation induced by ψ^*, $\mathbf{v}^i\big(\cdot;\theta^{-i},\psi^*\big)$, and the definition of the aggregator function ζ^i. This last expression shows that for any θ^i and $\hat{\theta}^i$, agent i's ex post incentive constraints are satisfied, as desired. ■

As before, we provide an example to illustrate our approach.

Example 4. We preserve the main features of previous examples. There are three agents in the economy, only agent 1 generates informational externalities. The allocation set is $\mathcal{X} = \mathbb{R}^3_+$; agent j's type space is $\Theta^j = [0,\overline{\theta}^j_1]\times[0,\overline{\theta}^j_2]$, for $j \in I$. Preferences for public alternatives are given by:

$$\begin{aligned}
\widetilde{V}^1(x,\theta^1) \;&=\; \theta^1_1\{\alpha^1_1 x_1 + \gamma^1 x_3\} \;+\; \theta^1_2\{\beta^1_2 x_2 + \gamma^1 x_3\};\\
\widetilde{V}^i(x,\theta^1,\theta^i) \;&=\; \theta^i_1\{\theta^1_1\alpha^i_1 x_1 + (\theta^1_1+\theta^1_2)\gamma^i x_3\} \;+\; \theta^i_2\{\theta^1_2\beta^i_2 x_2 + (\theta^1_1+\theta^1_2)\gamma^i x_3\}, \qquad i = 2,3.
\end{aligned}$$

The central planner selects an efficient allocation rule to maximize $\sum \widetilde{V}^j(x,\theta) - C(x)$, where the cost function is given by $C(x) = \frac{1}{2}\sum_{m=1}^3 x_m^2$. In this example, the first two component functions of the efficient allocation rule ψ^* are separable. The third component function depends on both dimensions of information, but in an additive way. Write $\zeta^i(\theta^i) = \theta^i_1 + \theta^i_2$; the efficient allocation rule $\psi^* = (\psi^*_1,\psi^*_2,\psi^*_3)$ is defined by

$$\psi^*_1[\theta_1] \;=\; \theta^1_1\{\alpha^1_1 + \alpha^2_1\theta^2_1 + \alpha^3_1\theta^3_1\}; \tag{24}$$

$$\psi^*_2[\theta_2] \;=\; \theta^1_2\{\beta^1_2 + \beta^2_2\theta^2_2 + \beta^3_2\theta^3_2\}; \tag{25}$$

$$\psi^*_3[\zeta(\theta)] \;=\; \zeta^1(\theta^1)\{\gamma^1 + \gamma^2\zeta^2(\theta^2) + \gamma^3\zeta^3(\theta^3)\}. \tag{26}$$

Using expression (23), agent 1's surplus extraction tax, when he reports $\hat{\theta}^1$ and agents 2 and 3 are truthfully reporting their types θ^2, θ^3, is:

$$\begin{aligned}\tau^1[\hat{\theta}^1,\theta^{-1}] &= \hat{\theta}_1^1\alpha_1^1\psi_1^*[\hat{\theta}_1^1,\theta_1^{-1}] - \int_0^{\hat{\theta}_1^1}\alpha_1^1\psi_1^*[\tilde{\theta}_1^1,\theta_1^{-1}]\,\mathrm{d}\tilde{\theta}_1^1 \\ &+ \hat{\theta}_2^1\beta_2^1\psi_2^*[\hat{\theta}_2^1,\theta_2^{-1}] - \int_0^{\hat{\theta}_2^1}\beta_2^1\psi_2^*[\tilde{\theta}_2^1,\theta_2^{-1}]\,\mathrm{d}\tilde{\theta}_2^1 \\ &+ \zeta^1(\hat{\theta}^1)\gamma^1\psi_3^*[\zeta^1(\hat{\theta}^1),\zeta^{-1}(\theta^{-1})] - \int_0^{\hat{\theta}_1^1+\hat{\theta}_2^1}\gamma^1\psi_3^*[\tilde{\zeta}^1,\zeta^{-1}(\theta^{-1})]\,\mathrm{d}\tilde{\zeta}^1.\end{aligned}$$

We replace the efficient allocation rule given in (24) to (26) in the above equation, and integrate to recover agent 1' incentive tax. After some manipulations, we find

$$\begin{aligned}\tau^1[\hat{\theta}^1,\theta^{-1}] &= \frac{1}{2}\alpha_1^1\left(\hat{\theta}_1^1\right)^2\left\{\alpha_1^1+\alpha_1^2\theta_1^2+\alpha_1^3\theta_1^3\right\} + \frac{1}{2}\beta_2^1\left(\hat{\theta}_2^1\right)^2\left\{\beta_2^1+\beta_2^2\theta_2^2+\beta_2^3\theta_2^3\right\} \\ &+ \frac{1}{2}\gamma^1\left(\hat{\theta}_1^1+\hat{\theta}_2^1\right)^2\left\{\gamma^1+\gamma^2(\theta_1^2+\theta_2^2)+\gamma^3(\theta_1^3+\theta_2^3)\right\}.\end{aligned}$$

To simplify notation, let $\kappa_1=\{\alpha_1^1+\alpha_1^2\theta_1^2+\alpha_1^3\theta_1^3\}$, $\kappa_2=\{\beta_2^1+\beta_2^2\theta_2^2+\beta_2^3\theta_2^3\}$ and $\kappa_3=\{\gamma^1+\gamma^2(\theta_1^2+\theta_2^2)+\gamma^3(\theta_1^3+\theta_2^3)\}$. Agent 1's utility when his type is θ^1 and his report $\hat{\theta}^1$, and agents 2 and 3 are truthfully reporting, is

$$\begin{aligned}\widetilde{V}^1\left(\psi^*[\hat{\theta}^1,\theta^{-1}],\theta\right) + \tau^1[\hat{\theta}^1,\theta^{-1}] &= \theta_1^1\alpha_1^1\hat{\theta}_1^1\kappa_1 + \theta_2^1\beta_2^1\hat{\theta}_2^1\kappa_2 + (\theta_1^1+\theta_2^1)\gamma^1(\hat{\theta}_1^1+\hat{\theta}_2^1)\kappa_3 \\ &- \frac{1}{2}\alpha_1^1(\hat{\theta}_1^1)^2\kappa_1 - \frac{1}{2}\beta_2^1(\hat{\theta}_2^1)^2\kappa_2 - \frac{1}{2}\gamma^1(\hat{\theta}_1^1+\hat{\theta}_2^1)^2\kappa_3.\end{aligned}$$

One can easily verify that the optimal report for agent 1 is $\hat{\theta}_1^1=\theta_1^1$ and $\hat{\theta}_2^1=\theta_2^1$. □

We finish this subsection putting our positive result in perspective with the impossibility result of Jehiel, Meyer-ter-Vehn, Moldovanu, and Zame (2006), whose work has shown that, for generic valuation functions, the only allocation rules that are ex post implementable are constant rules. Jehiel et al consider a general social choice setup with 2 agents and 2 alternatives, and use the ex post taxation principle[20] to express i's tax burden in terms of agent j's multi-dimensional type and the alternative selected according to the given allocation rule. The advantage of taxation principle is that it explicitly shows that in an ex post incentive compatible mechanism, the selection of the allocation rule coincides with the individual choices, which are aligned using the tax system. Therefore, the social indifference set (a subset of the information space for which both alternatives are selected) coincides with the individual indifference sets (a subset of the information space for which an agent is indifferent between either of the alternatives). These common indifference sets are multi-dimensional submanifolds of $\Theta^1\times\Theta^2$. The impossibility result relies on showing that a geometric condition relating agents' rates of information substitution, which must hold everywhere in the

[20]See Chung and Ely (2002).

common indifference sets if the ex post incentive constraints are to be satisfied, is not satisfied by generic valuations.

We argue that this powerful impossibility result is overcome in quasi-separable environments. Indeed, our Assumption 3 implies, first, that the efficient allocation rule is quasi-separable, and second, that every dimension of information has equal marginal impact on the appraisal of x_{K+1}. Fix a type profile for all agents other than i, and consider a marginal increase in the reported θ^i_k. This variation affects the chosen social alternative in its k and its $K+1$ components, in a way that cannot be compensated by a marginal variation in, say, θ^i_l (for $l \neq k$). Suppose we try otherwise with a marginal decrease in the reported θ^i_l. Since both dimensions of preference heterogeneity affect the appraisal of x_{K+1} equally, this project returns to its original level. But x_k and x_l will not vary in the same way, unless preferences for projects k and l are identical among all agents (in which case k and l are but one project). Therefore, the social indifference set at θ has dimension zero, and the geometric condition of Jehiel, Meyer-ter-Vehn, Moldovanu, and Zame (2006) can be satisfied (since it needs to hold only at one point).

4 Concluding remarks

In this paper, we have presented some positive results for efficient, ex post mechanism design with interdependent valuations. We first provided a complete characterization of ex post implementable allocation rules when valuations are linear in individual types, from which an ex post payoff equivalent principle was derived. We have shown that if types are one-dimensional, then the surplus extraction tax scheme recovered from first order incentive compatibility constraints implements the efficient allocation rule as an ex post equilibrium. This incentive tax scheme is equivalent, up to an additive constant, to the generalized VCG tax scheme presented elsewhere in the literature (e.g., Bergemann and Välimäki (2002)). Moreover, if types are multi-dimensional, the surplus extraction tax scheme can be adapted to implement the efficient allocation rule in separable and quasi-separable environments. We have also argued that the geometric condition for the impossibility result of Jehiel, Meyer-ter-Vehn, Moldovanu, and Zame (2006) does not bite in the separable and quasi-separable case.

We consider several lines of further research worth pursuing. First, one can relax the linearity in types assumption of valuation functions. Although this relaxation will weaken the characterization of ex post implementable allocation rules, one should still be able to recover a surplus extraction tax scheme from the system of differential equations describing first order conditions of incentive constraints, under separable and quasi-separable environments.

Feasibility of the mechanism is important in some economic applications, e.g., the classic public good provision problem. In others, imposing a certain type of participation constraints, which may be endogenous, seems more appropriate. One could investigate under which conditions there are efficient, ex post mechanisms satisfying these, and other, desirable properties. To do so, it shall suffice to focus the attention on surplus extraction taxes and examine what properties do the constants of integration satisfy.

Appendix

Proof of Proposition 1. For necessity, suppose that the allocation rule ψ is ex post implementable by the transfer scheme τ. Then, for every θ^{-i} in Θ^{-i}, the indirect utility function $\theta^i \to U^i\big(\theta^i;\theta^{-i},\Gamma\big)$ is convex on Θ^i. As it has been argued,

$$\mathbf{v}^i\big(\theta^i;\theta^{-i},\psi\big) \quad \in \quad \partial U^i\big(\theta^i;\theta^{-i},\Gamma\big)\ , \qquad \forall\ \theta^i \in \Theta^i;$$

and thus condition (i) follows readily from Lemma 1. Let θ^i and $\hat{\theta}^i$ belong to Θ^i; using the definition of the subdifferential of the convex function $U^i\big(\cdot;\theta^{-i},\Gamma\big)$ at θ^i and $\hat{\theta}^i$, respectively, we see that

$$U^i\big(\hat{\theta}^i;\theta^{-i},\Gamma\big) \quad \geq \quad U^i\big(\theta^i;\theta^{-i},\Gamma\big) \ + \ \mathbf{v}^i\big(\theta^i;\theta^{-i},\psi\big)\cdot\big\{\hat{\theta}^i-\theta^i\big\},$$

and

$$U^i\big(\theta^i;\theta^{-i},\Gamma\big) \quad \geq \quad U^i\big(\hat{\theta}^i;\theta^{-i},\Gamma\big) \ + \ \mathbf{v}^i\big(\hat{\theta}^i;\theta^{-i},\psi\big)\cdot\big\{\theta^i-\hat{\theta}^i\big\}.$$

Adding up the previous two equations, we obtain condition (ii).

Conversely, suppose that (i) and (ii) are satisfied. We must show that the allocation rule ψ is ex post implementable by some tax scheme τ. That is, we must show that for every $i=1,\ldots,I$, every θ^{-i} in Θ^{-i} and every θ^i in Θ^i, expression (4) is satisfied. Agent i's payoff differential from reporting θ^i and $\hat{\theta}^i$, when i's true type is θ^i, is

$$\begin{aligned} U^i\big(\theta^i;\theta^{-i},\Gamma\big) \ &- \ \overline{U}^i\big(\hat{\theta}^i,\theta^i;\theta^{-i},\Gamma\big) \\ &= \ U^i\big(\theta^i;\theta^{-i},\Gamma\big) \ - \ \big\{V^i\big(\psi[\hat{\theta}^i,\theta^{-i}],\hat{\theta}^i,\theta^{-i}\big) \ - \ \tau^i[\hat{\theta}^i,\theta^{-i}]\big\} \\ &\quad - \ V^i\big(\psi[\hat{\theta}^i,\theta^{-i}],\theta^i,\theta^{-i}\big) \ + \ V^i\big(\psi[\hat{\theta}^i,\theta^{-i}],\hat{\theta}^i,\theta^{-i}\big) \\ &= \ U^i\big(\theta^i;\theta^{-i},\Gamma\big) \ - \ U^i\big(\hat{\theta}^i;\theta^{-i},\Gamma\big) \ - \ \mathbf{v}^i\big(\hat{\theta}^i;\theta^{-i},\psi\big)\cdot\big\{\theta^i-\hat{\theta}^i\big\}. \end{aligned}$$

Therefore, the allocation rule ψ will be ex post implementable provided the following holds, for any two θ^i, $\hat{\theta}^i$ in Θ^i:

$$(27) \qquad U^i\big(\theta^i;\theta^{-i},\Gamma\big) \ - \ U^i\big(\hat{\theta}^i;\theta^{-i},\Gamma\big) \ - \ \mathbf{v}^i\big(\hat{\theta}^i;\theta^{-i},\psi\big)\cdot\big\{\theta^i-\hat{\theta}^i\big\} \quad \geq \quad 0.$$

Replacing condition (i) in equation (27), one obtains

$$\int_0^1 \big\{\mathbf{v}^i\big(\theta^i(\lambda);\theta^{-i},\psi\big)-\mathbf{v}^i\big(\hat{\theta}^i;\theta^{-i},\psi\big)\big\}\cdot\big\{\theta^i-\hat{\theta}^i\big\}\ \mathrm{d}\lambda \quad \geq \quad 0;$$

where as before λ lies in the open unit interval, and $\theta^i(\lambda)=\lambda\theta^i+(1-\lambda)\hat{\theta}^i$. Observe now that $\theta^i \ - \ \hat{\theta}^i = \frac{1}{\lambda}[\theta^i(\lambda)-\hat{\theta}^i]$. Hence, for any $\lambda\in(0,1)$ we have

$$\begin{aligned} \big\{\mathbf{v}^i\big(\theta^i(\lambda);\theta^{-i},\psi\big) \quad &- \quad \mathbf{v}^i\big(\hat{\theta}^i;\theta^{-i},\psi\big)\big\}\cdot\big\{\theta^i-\hat{\theta}^i\big\} \\ &= \quad \frac{1}{\lambda}\big\{\mathbf{v}^i\big(\theta^i(\lambda);\theta^{-i},\psi\big) \ - \ \mathbf{v}^i\big(\hat{\theta}^i;\theta^{-i},\psi\big)\big\}\cdot\big\{\theta^i(\lambda)-\hat{\theta}^i\big\} \\ &\geq \quad 0 \end{aligned}$$

with the last inequality derived by condition (ii). This implies that (27) is satisfied. ■

References

AUSUBEL, L. M. (2004): "An efficient ascending-bid auction for multiple objects," *American Economic Review*, 94(5), 1452–1475.

BERGEMANN, D., AND S. MORRIS (2005): "Robust mechanism design," *Econometrica*, 73(6), 1771–1813.

BERGEMANN, D., AND J. VÄLIMÄKI (2002): "Information acquisition and efficient mechanism design," *Econometrica*, 70(3), 1007–11033.

CHUNG, K.-S., AND J. C. ELY (2002): "Ex-post incentive compatible mechanism design," Mimeo: Department of Economics, Northwestern University.

CLARKE, E. H. (1971): "Multipart pricing of public goods," *Public Choice*, 11(1), 17–33.

DASGUPTA, P., AND E. MASKIN (2000): "Efficient auctions," *Quarterly Journal of Economics*, 115(2), 341–388.

FANG, H., AND P. NORMAN (2006): "Optimal provision of multiple excludable public goods," Mimeo: Yale University and University of British Columbia.

GREEN, J. R., AND J.-J. LAFFONT (1979): *Incentives in Public Decision-Making*, Studies in Public Economics. North-Holland, New York, NY.

GROVES, T. (1973): "Incentives in teams," *Econometrica*, 41(4), 617–631.

HIRIART-URRUTY, J.-B., AND C. LEMARÉCHAL (2001): *Fundamentals of Convex Analysis*, Grundlehren Text Editions. Springer-Verlag, Berlin, D.

HOWARD, R. (1998): "Alexandrov's theorem on the second derivative of convex functions," Mimeo: Department of Mathematics, University of South Carolina, Columbia S.C.

JEHIEL, P., M. MEYER-TER-VEHN, B. MOLDOVANU, AND W. R. ZAME (2006): "The limits of ex post implementation," *Econometrica*, 74(3), 585–610.

JEHIEL, P., AND B. MOLDOVANU (2001): "Efficient mechanism design with interdependent valuations," *Econometrica*, 69(5), 1237–1259.

——— (2004): "The design of an efficient private industry," *Journal of the European Economic Association*, 2, 516–525.

——— (2006): "Allocative and informational externalities in auctions and related mechanisms," in *Advances in Economics and Econometrics: Theory and Applications, Ninth World Congress*, ed. by R. Blundell, W. K. Newey, and T. Persson, chap. 3. Cambridge University Press, Cambridge, UK.

KRISHNA, V., AND E. MAENNER (2001): "Convex potentials with an application to mechanism design," *Econometrica*, 69(4), 1113–1119.

Laffont, J.-J., and E. Maskin (1980): "A differential approach to dominant strategy mechanisms," *Econometrica*, 48(6), 1507–1520.

Ledyard, J. O., and T. R. Palfrey (2007): "A general characterization of interim efficient mechanisms for independent linear environments," *Journal of Economic Theory*, 133(1), 441–466.

Manelli, A. M., and D. R. Vincent (2006): "Bundling as an optimal selling mechanism for a multiple-good monopolist," *Journal of Economic Theory*, 127(1), 1–35.

McLean, R. P., and A. Postlewaite (2006): "Implementation with interdependent valuations," Mimeo: Departments of Economics, University of Pennsylvania.

Myerson, R. B. (1981): "Optimal auction design," *Mathematics of Operations Research*, 6(1), 58–73.

Perry, M., and P. J. Reny (2002): "An efficient auction," *Econometrica*, 70(3), 1199–1212.

Rochet, J.-C. (1987): "A necessary and sufficient condition for rationalizability in a quasi-linear context," *Journal of Mathematical Economics*, 16(2), 191–200.

Rochet, J.-C., and P. Choné (1998): "Ironing, sweeping, and multidimensional screening," *Econometrica*, 66(4), 783–826.

Rochet, J.-C., and L. A. Stole (2003): "The economics of multidimensional screening," in *Advances in Economics and Econometrics: Theory and Applications. Eighth World Congress of the Econometric Society*, chap. 5. Cambridge University Press, Cambridge, UK.

Vickrey, W. (1961): "Counterspeculation, auctions, and competitive sealed tenders," *Journal of Finance*, 16(1), 8–37.