Efficient provision of multiple public goods with interdependent valuations

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Introduction

Consider a public good provision problem with:

- multiple public goods, $x = (x_1, \ldots, x_M)$;
- I agents with quasi-linear preferences;
- multi-dimensional types;
- interdependent valuations: $\mathcal{U}_i(x, \theta, t_i) = \mathcal{V}_i(x, \theta) - t_i$.

Last two features are present in many economic situations:

- oil field auction;
- road project that affects competing firms;
- investments in an educational institution.

Introduction

Mechanism Design approach:

- Center uses $\Gamma = (\psi, \tau)$ to select social outcome.
  - ψ is the allocation rule;
  - $\tau = (\tau_1, \ldots, \tau_I)$ is the tax scheme.
- $\Gamma = (\psi, \tau)$ is *ex post incentive compatible* if truth-telling constitutes an ex post equilibrium.

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JMMZ (2006) impossibility result:

- In a 2-agent, 2-alternative setting, it is generally impossible to implement non-trivial allocation rules.
- Geometric condition derived from taxation principle.
  - Choices under ψ coincide with individual choices.

Contribution of this paper

- Solution to the public good provision problem with interdependent valuations and multi-dimensional types.
- Two environments where efficient, ex post incentive compatible mechanisms exist.
- Environments are non-generic but important.

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Outline:

- General linear setup for public good provision problem.
- Characterize ex post implementable allocation rules.
- Efficient solution for one-dimensional environments.
- Extension to separable and quasi-separable environments.

General Linear Setup

- Multiple public goods: $\mathcal{X} = \mathbb{R}^M_+$; $x = (x_1, \ldots, x_M)$.
- Cost of provision is $C(x) = \sum_m c_m(x_m)$.
- I agents with quasi-linear preferences and multidimensional types: $\theta_i = (\theta_i^1, \ldots, \theta_i^K) \in \Theta_i = \times_{k=1}^K [0, \overline{\theta}_i^k]$.
- Preferences are linear in individual types:

$$\mathcal{U}_i(x, \theta, t_i) \;=\; \sum_1^K \theta_i^k v_i^k(x, \theta_{-i}) - t_i \;=:\; \mathcal{V}_i(x, \theta) - t_i.$$

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Standing hypothesis:

- $\forall\ \theta$, there is a unique $\psi^*[\theta] \in \mathcal{X}$ that maximizes social welfare; $\psi^* : \Theta \to \mathcal{X}$ is continuously differentiable on Θ.

Application: educational institution

- $x = (x_1, x_2, x_3)$ is composed of $M = 3$ different projects.
  - x_1 size of the science facility;
  - x_2 size of the humanities facility;
  - x_3 size of the central library.
- Agents have $K = 2$ dimensions of private information.
  - θ_i^1 appreciation for science and mathematics;
  - θ_i^2 appreciation for arts and humanities.
- Center selects the efficient allocation and the division of the tax burden.
  - Efficient allocation rule ψ^* exists.
  - Center designs incentive tax scheme to induce honest reporting as ex post equilibrium.

Characterization: ex post allocation rules

Given ψ and $\theta_{-i} \in \Theta_{-i}$, use i's indirect utility function U_i.

- U_i is the max-function of i's ex post incentive problem [the ex post self-selection constraints].
- U_i is a convex function.
- In the linear setup, ex post implementation of allocation rule ψ is characterized by:
  (i) integral representation of U_i;
  (ii) a monotonicity condition.
- From this characterization, it follows the ex post payoff equivalence principle for linear environments.

Characterization: ex post allocation rules

- ψ is ex post implementable by an incentive tax scheme $\tau = (\tau_1, \ldots, \tau_I)$ if and only if $\forall\, i$, $\forall\, \theta_{-i} \in \Theta_{-i}$:

(i) For any two θ_i and $\hat{\theta}_i$,

$$U_i(\theta_i) \;=\; U_i(\hat{\theta}_i) + \int_0^1 \mathbf{v}_i\big(\psi[\theta_i(\lambda), \theta_{-i}], \theta_{-i}\big) \cdot \big\{\theta_i - \hat{\theta}_i\big\}\, \mathrm{d}\lambda;$$

where $\lambda \in (0,1)$ and $\theta_i(\lambda) := \lambda\theta_i + (1-\lambda)\hat{\theta}_i$.

(ii) The reduced form allocation $\mathbf{v}_i\big(\psi[\cdot, \theta_{-i}], \theta_{-i}\big)$ is increasing; i.e., for any two θ_i and $\hat{\theta}_i$,

$$\big\{\mathbf{v}_i\big(\psi[\theta_i, \theta_{-i}], \theta_{-i}\big) - \mathbf{v}_i\big(\psi[\hat{\theta}_i, \theta_{-i}], \theta_{-i}\big)\big\} \cdot \big\{\theta_i - \hat{\theta}_i\big\} \;\geq\; 0.$$

Necessary condition

From the characterization result, derive a necessary condition for ex post implementation.

- Related to solvability of a system of differential equations from the multi-dimensional IC constraints.
- Indirect utility U_i is twice differentiable a.e. on Θ_i.
- ψ is ex post implementable only if $\forall\ \theta_{-i}$, for any two dimensions of private information $k \neq l$, a.e. on Θ_i:

$$\frac{\partial^2 U_i(\theta_i)}{\partial\theta_i^l\,\partial\theta_i^k} = \frac{\partial^2 U_i(\theta_i)}{\partial\theta_i^k\,\partial\theta_i^l}. \tag{1}$$

- Necessary condition is not generally satisfied.

Example

- $\mathcal{X} = \mathbb{R}^3_+; \quad C(x) = \frac{1}{2}\sum_{m=1}^3 (x_m)^2$.
- $I = \{1,2,3\}; \quad \Theta_i = [0, \overline{\theta}_i^1] \times [0, \overline{\theta}_i^2]$.
- Valuations for the public good are:

$$\begin{aligned}
\mathcal{V}_1(x,\theta) &= \theta_1^1\{\alpha_1^1 x_1 + \beta_1^1 x_2 + \gamma_1^1 x_3\} + \theta_1^2\{\alpha_1^2 x_1 + \beta_1^2 x_2 + \gamma_1^2 x_3\}; \\
\mathcal{V}_i(x,\theta) &= \theta_i^1\{\theta_1^1(\alpha_i^1 x_1 + \beta_i^1 x_2 + \gamma_i^1 x_3)\} \\
&\quad + \theta_i^2\{\theta_1^2(\alpha_i^2 x_1 + \beta_i^2 x_2 + \gamma_i^2 x_3)\}, \qquad i = 2,3.
\end{aligned}$$

- Given θ, Center chooses the allocation that maximizes social welfare $\sum_i \mathcal{V}_i(x,\theta) - C(x)$.

Example

- The efficient allocation rule $\psi^* = \left(\psi_1^*, \psi_2^*, \psi_3^*\right)$ is

$$\begin{aligned}
\psi_1^*[\theta] &= \theta_1^1\left(\alpha_1^1 + \alpha_2^1\theta_2^1 + \alpha_3^1\theta_3^1\right) \;+\; \theta_1^2\left(\alpha_1^2 + \alpha_2^2\theta_2^2 + \alpha_3^2\theta_3^2\right), \\
\psi_2^*[\theta] &= \theta_1^1\left(\beta_1^1 + \beta_2^1\theta_2^1 + \beta_3^1\theta_3^1\right) \;+\; \theta_1^2\left(\beta_1^2 + \beta_2^2\theta_2^2 + \beta_3^2\theta_3^2\right), \\
\psi_3^*[\theta] &= \theta_1^1\left(\gamma_1^1 + \gamma_2^1\theta_2^1 + \gamma_3^1\theta_3^1\right) \;+\; \theta_1^2\left(\gamma_1^2 + \gamma_2^2\theta_2^2 + \gamma_3^2\theta_3^2\right).
\end{aligned}$$

- Use expression (1) to verify if ψ^* can be implemented ex post. For any θ_2 and θ_3:

$$\begin{aligned}
&\alpha_1^1\left(\alpha_2^2\theta_2^2 + \alpha_3^2\theta_3^2\right) \;+\; \beta_1^1\left(\beta_2^2\theta_2^2 + \beta_3^2\theta_3^2\right) \;+\; \gamma_1^1\left(\gamma_2^2\theta_2^2 + \gamma_3^2\theta|_3^2\right) \\
&\quad = \;\alpha_1^2\left(\alpha_2^1\theta_2^1 + \alpha_3^1\theta_3^1\right) \;+\; \beta_1^2\left(\beta_2^1\theta_2^1 + \beta_3^1\theta_3^1\right) \;+\; \gamma_1^2\left(\gamma_2^1\theta_2^1 + \gamma_3^1\theta_3^1\right).
\end{aligned}$$

One-dimensional environments

- $\Theta_i = [0, \overline{\theta}_i]$; $\quad \mathcal{U}_i(x, \theta, t_i) = \theta_i v_i(x, \theta_{-i}) - t_i$.
- Allocation set is one-dimensional: $\mathcal{X} = \mathbb{R}_+$.
- Necessary condition for ex post implementation is trivially satisfied.
- Use characterization result to construct tax scheme that implements ψ^*.
- $\Gamma = (\psi^*, \tau^*)$ such that

$$\tau_i^*[\theta] \quad = \quad \theta_i v_i(\psi^*[\theta], \theta_{-i}) - \int_0^{\theta_i} \mathbf{v}_i(\psi^*[\tilde{\theta}_i, \theta_{-i}], \theta_{-i}) \, \mathrm{d}\tilde{\theta}_i.$$

Separable environments

- Dimension of public good space equals the dimension of information space, $M = K$.
  - $\Theta_i = \times_{k=1}^{K} [0, \overline{\theta}_i^k]; \quad \mathcal{X} = \mathbb{R}_+^K$.
- Valuations are separable:

$$\mathcal{U}_i(x, \theta, t_i) \;=\; \sum_k \theta_i^k \, \widehat{v}_i^k(x_k, \theta_{-i}^k) \;-\; t_i.$$

- Each dimension of information is specific to one, and only one, project.
  - Multi-dimensional screening: Rochet&Chone (1998).
  - Bundling with private goods: Manelli&Vincent (2006).
  - Bundling with public goods: Fang&Norman (2007).

Separable environments

- Efficient allocation rule is separable: project k depends on k-th dimension of information:

$$\psi^*[\theta] \quad = \quad \left(\psi_1^*[\theta^1], \ldots, \psi_K^*[\theta^K]\right).$$

- Necessary condition for ex post implementation of ψ^* is satisfied.

Separable environments

- Efficient allocation rule is separable: project k depends on k-th dimension of information:

$$\psi^*[\theta] \quad = \quad \left(\psi_1^*[\theta^1], \ldots, \psi_K^*[\theta^K]\right).$$

- Left-hand side of equation (1) is:

$$\frac{\partial^2 U_i(\theta_i)}{\partial \theta_i^l \, \partial \theta_i^k} \quad = \quad \frac{\partial \widehat{v}_i^k}{\partial x_k}(\psi_k^*[\theta^k], \theta_{-i}^k) \, \frac{\partial \psi_k^*}{\partial \theta_i^l}[\theta^k] \quad = \quad 0.$$

Separable environments

- Efficient allocation rule is separable: project k depends on k-th dimension of information:

$$\psi^*[\theta] \quad = \quad \left(\psi_1^*[\theta^1], \ldots, \psi_K^*[\theta^K]\right).$$

- Necessary condition for ex post implementation of ψ^* is satisfied.
- Multi-dimensional public good provision problem is transformed into K one-dimensional problems.
- ψ^* is implemented by a separable tax scheme. $\Gamma = (\psi^*, \tau^*)$ such that

$$\tau_i^*[\theta] \quad = \quad \sum\nolimits_{k=1}^{K} \tau_i^{k*}[\theta^k].$$

Quasi-separable environments

- Aggregator function: $\zeta_i(\theta_i) = \theta_i^1 + \ldots + \theta_i^K$.
- Dimension of $\mathcal{X}$ is $M = K + 1$; $x = \big(x_1, \ldots, x_K, x_{K+1}\big)$.
- Valuations satisfy:

$$\begin{aligned}\mathcal{U}_i(x, \theta, t_i) &= \sum_{k=1}^{K} \theta_i^k \, \widetilde{v}_i^k(x_k, \theta_{-i}^k) \\ &\quad + \zeta_i(\theta_i)\, \widetilde{w}_i\big(x_{K+1}, \zeta_{-i}(\theta_{-i})\big) - t_i.\end{aligned}$$

- k-th dimension of information affects project k.
- k-th dimension of information affects project $K + 1$.
  This interaction is uniformed across dimensions.

Quasi-separable environments

- Efficient allocation rule is quasi-separable:

$$\psi^*[\theta] \quad = \quad \left(\psi_1^*[\theta^1], \ldots, \psi_K^*[\theta^K], \psi_{K+1}^*[\zeta(\theta)]\right).$$

- Necessary condition for ex post implementation of ψ^* is satisfied.

Quasi-separable environments

- Efficient allocation rule is quasi-separable:

$$\psi^*[\theta] \quad = \quad \left(\psi_1^*[\theta^1], \ldots, \psi_K^*[\theta^K], \psi_{K+1}^*[\zeta(\theta)]\right).$$

- Equation (1) is satisfied; for $k \neq l$:

$$\begin{aligned} \frac{\partial^2 U_i(\theta_i)}{\partial \theta_i^l \, \partial \theta_i^k} &= \left(0, \ldots, \frac{\partial \widetilde{v}_i^k}{\partial x_k}, 0, \ldots, \frac{\partial \widetilde{w}_i}{\partial x_{K+1}}\right) \cdot \left(0, \ldots, 0, \frac{\partial \psi_l^*}{\partial \theta_i^l}, \ldots, \frac{\partial \psi_{K+1}^*}{\partial \zeta_i} \frac{\partial \zeta_i}{\partial \theta_i^l}\right) \\ \frac{\partial^2 U_i(\theta_i)}{\partial \theta_i^k \, \partial \theta_i^l} &= \left(0, \ldots, 0, \frac{\partial \widetilde{v}_i^l}{\partial x_l}, \ldots, \frac{\partial \widetilde{w}_i}{\partial x_{K+1}}\right) \cdot \left(0, \ldots, \frac{\partial \psi_k^*}{\partial \theta_i^k}, 0, \ldots, \frac{\partial \psi_{K+1}^*}{\partial \zeta_i} \frac{\partial \zeta_i}{\partial \theta_i^k}\right). \end{aligned}$$

Quasi-separable environments

- Efficient allocation rule is quasi-separable:

$$\psi^*[\theta] \quad = \quad \left(\psi_1^*[\theta^1], \ldots, \psi_K^*[\theta^K], \psi_{K+1}^*[\zeta(\theta)]\right).$$

- Necessary condition for ex post implementation of ψ^* is satisfied.
- ψ^* ex post implemented by quasi-separable incentive tax scheme. $\Gamma = (\psi^*, \tau^*)$ such that

$$\tau_i^*[\theta] \quad = \quad \sum_{k=1}^K \tau_i^{k*}[\theta^k] \quad + \quad \tau_i^{K+1*}[\zeta(\theta)].$$

Quasi-separable environments

The efficient provision of multiple public goods is consistent with geometric condition of JMMZ (2006).

- From taxation principle, indifference sets of all agents coincide with indifference set generated by ψ^*.
- $\mathcal{I}(\theta_{-i}) \subseteq \Theta_i$ is indifference set with fixed θ_{-i}.
- Start at $\theta_i \in \mathcal{I}(\theta_{-i})$:
  - $\triangle\theta_i^k$ affects x_k and x_{K+1};
  - $\bigtriangledown\theta_i^l$ affects x_l and x_{K+1};
  - changes in x_{K+1} cancel out.
- At any θ_{-i}, indifference set $\mathcal{I}(\theta_{-i})$ has dimension zero.

Concluding remarks

- With interdependent valuations, efficient ex post mechanisms for the provision of multiple public goods exist in separable and quasi-separable environments.
  - Impossibility result of JMMZ (2006) does not apply.
  - These environments are economically important.
- Other environments where positive results exist.
  - JMM (2007) use separability across valuations.
- Directions for further research:
  - Relax linearity assumption on valuations.
  - Existence of efficient ex post mechanisms that are budget balanced, individually rational, etc.